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An Algorithm for Computing Spline Function

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Abstract

A numerical algorithm is constructed to develop numerical solution to the spline function based belonging to the C^6-class. The presented method showed that the approximate solution for boundary value problems obtain by the numerical algorithm which are applied sixtic spline function is effective. Convergence analysis of the proposed method and error estimates are obtained. Numerical results illustrate by two examples are given the practical usefulness and efficiency of the algorithm.

Introduction

In recent years, the spline interpolations have been considered in many ways gained popularity in some field (e.g. initial value problems IVPs and boundary value problems BVPs), see [1-7].

Fawzy [8] constructed the lacunary spline model based on quartic polynomial, they obtained existence and uniqueness with convergence analysis of the model. In [4, 5] some types of fractional spline function used to derive some error bounds which are used to develop a numerical algorithm for solving differential equations of fractional order. In (2012) Srivastava et al [9] developed the numerical algorithm for quantic non-polynomial spline for approximate solution to certain boundary value problems involving the third-order ordinary differential equation associated with draining and coating flow. Amongst further workers on this subject list as different kinds of spline degree have been available for decades [10, 11, 12], they have mainly been utilized for analysis convergence and recently as interpolation polynomials for solving differential equations [1, 7, 13, 14].

The main objective of the present paper is to apply a sixtic integer degree spline algorithm (see [4, 5, and 8]) with a polynomial part to develop a numerical method for obtaining smooth approximations of the solution of ordinary differential equations.

Sixtic Spline Formulation

In order to develop the numerical algorithm by sixtic spline interpolation function $S_\Delta(x) \in C^6$, interpolating to a function $y(x)$ defined on [a, b], which are developed from Faraidun and Fawzy [4, 8] respectively, we know that $y(x)$ at the mesh point in each subintervals$[x_{k-1}, x_k]$, the spline function $S_\Delta(x)$ of degree at most six. We construct the spline functions of degree six in the closed interval [0, 1] containing a parameters are a_k, b_k and c_k.

Theorem 2.1 Let $S_\Delta(x)$ in [0, 1] be a spline function define as

$$S_k(x) = y_k + a_k(x - x_k) + \frac{1}{2}y_k''(x - x_k)^2 + \frac{1}{6}b_k(x - x_k)^3 + \frac{1}{24}y_k^{(4)}(x - x_k)^4 + \frac{1}{120}c_k(x - x_k)^5 + \frac{1}{720}y_k^{(6)}(x - x_k)^6 , \quad (1)$$

where $x_k < x < x_{k+1}$ and k= 0, 1, 2,…, n-1 . Then there exist and unique spline interpolation function $S_\Delta(x)$.

Proof: From equation (1), and using Taylor series expansion, we get

$$y_{k+1} = y_k + a_k(x_{k+1} - x_k) + \frac{1}{2}y_k''(x_{k+1} - x_k)^2 + \frac{1}{6}b_k(x_{k+1} - x_k)^3 + \frac{1}{24}y_k^{(4)}(x_{k+1} - x_k)^4 + \frac{1}{120}c_k(x_{k+1} - x_k)^5 + \frac{1}{720}y_k^{(6)}(x_{k+1} - x_k)^6 . \quad (2)$$

Taking the second and fourth derivatives respectively for the boundary condition which known, we get

$$y''_{k+1} = y''_k + b_k(x_{k+1} - x_k) + \frac{1}{2}y_k^{(4)}(x_{k+1} - x_k)^2 + \frac{1}{6}c_k(x_{k+1} - x_k)^3 + \frac{1}{24}y_k^{(6)}(x_{k+1} - x_k)^4 \quad (3)$$

$$y_{k+1}^{(4)} = y_k^{(4)} + c_k(x_{k+1} - x_k) + \frac{1}{2}y_k^{(6)}(x_{k+1} - x_k)^2. \quad (4)$$

By solving the equations (2), (3) and (4) simultaneously, we can find the parameters a_k, b_k and c_k, and let $h = x_{k+1} - x_k$, we get:

$$c_k = \frac{1}{h}(y_{k+1}^{(4)} - y_k^{(4)}) - \frac{1}{2}hy_k^{(6)}, \quad (5)$$

$$b_k = \frac{1}{h}(y''_{k+1} - y''_k) - \frac{h}{6}(2y_k^{(4)} + y_{k+1}^{(4)}) + \frac{h^3}{24}y_k^{(6)}, \quad (6)$$

$$a_k = \frac{1}{h}(y_{k+1} - y_k) - \frac{h}{3}(2y''_k + y''_{k+1}) + \frac{h^3}{360}(4y_k^{(4)} + y_{k+1}^{(4)}) - \frac{h^5}{240}y_k^{(6)}. \quad (7)$$

Clearly, the spline function which defined in equation (1) is uniquely exist. Moreover, by construction it is clear that $S_\Delta(x)$ in [0, 1] is a spline interpolation polynomial of degree 6.

Main Theoretical Results

In this section, we prove the convergence analysis of the spline model in equation (1), and error bounds of it.

Lemma 3.1: let y(x) $\in C^6[0,1]$ then we have the following inequalities:

$|a_k - y_k'| \le \frac{h^5}{90}\omega_6(h), \left|b_k - y_k^{(3)}\right| \le \frac{h^3}{12}\omega_6(h)\ and\ |c_k - y_k^{(5)}| \le \frac{h}{2}\omega_6(h),$

where $\omega_6(h)$ is the modulus of continuity of $y^{(6)}(x)$.

Proof: from equation (7), we get

$|a_k - y_k'| = |\frac{1}{h}[y_{k+1} - y_k - \frac{1}{3}y_k''h^2 - \frac{1}{6}y_{k+1}''h^2 + \frac{1}{45}y_k^{(4)}h^4 + \frac{7}{360}y_{k+1}^{(4)}h^4 - \frac{3}{720}y_k^{(6)}h^6\ ] - y_k'|,$

Using Taylor series expansion to obtain

$|a_k - y_k'| \le \frac{h^5}{720}\left(5\omega_6(h) + 2\omega_6(h) + \omega_6(h)\right) = \frac{h^5}{90}\omega_6(h),$

$$|a_k - y_k'| \le \frac{h^5}{90}\omega_6(h). \tag{8}$$

Now for $|b_k - y_k^{(3)}|$, by using equation (6), we obtain

$|b_k - y_k^{(3)}| \le \frac{h^3}{24}[\left|y^{(6)}(\alpha_1)\ - y^{(6)}(\alpha_2)\right| + |y_k^{(6)} - y^{(6)}(\alpha_3)|],\ \ x_k < \alpha_1, \alpha_2, \alpha_3 < x_{k+1}$

$$\le \frac{h^3}{24}\left(\omega_6(h) + \omega_6(h)\right) = \frac{h^3}{12}\omega_6(h). \tag{9}$$

Similarly, we obtain

$$\left|c_k - y_k^{(5)}\right| \le \frac{h}{2}\omega_6(h), \tag{10}$$

Theorem 3.2 Let $S_k(x)$ be the spline function defined in equation (1), and interpolate $y(x) \in C^6[0,1]$, then the inequality for all $x \in [0,1]$

$$\left|S_k^{(i)}(x) - y^{(i)}(x)\right| \le c_i h^{6-i}\omega_6(h), \tag{11}$$

holds for all $i = 0,1,2,3,4$, where $c_0 = \frac{11}{360}, c_1 = \frac{59}{720}, c_2 = \frac{5}{24}, c_3 = \frac{1}{2}, c_4 = 1 and\ c_5 = \frac{3}{2}$.

Proof. To prove this theorem, we using the result of the previous Lemma 2.1

$|S_k(x) - y(x)| = |h(a_k -\ y_k') + \frac{h^3}{6}\left(b_k - y_k^{(3)}\right) + \frac{h^5}{120}(c_k - y_k^{(5)}) + \frac{h^6}{720}\left(y_k^{(6)} - y^{(6)}(n_k)\right)|,$ where $x_k < n_k < x_{k+1}$,

$|S_k(x) - y(x)| \le h\left[\frac{h^5}{90}\right]\omega_6(h) + \frac{h^3}{6}\left[\frac{h^3}{12}\right]\omega_6(h) + \frac{h^5}{120}\left[\frac{h}{2}\right]\omega_6(h) + \frac{h^6}{720}\omega_6(h)$

$= \frac{11}{360}h^6\omega_6(h),$

$|S_k'(x) - y'(x)| \le \frac{h^5}{90}\omega_6(h) + \frac{h^2}{2}\left(\frac{h^3}{12}\omega_6(h)\right) + \frac{h^4}{24}\left(\frac{h}{2}\omega_6(h)\right) + \frac{h^5}{120}\omega_6(h)$

$$= \frac{59}{720} h^5 \omega_6(h), \qquad \text{where } x_k < n_k < x_{k+1},$$

$$|S_k''(x) - y''(x)| \le h|b_k - y_k^{(3)}| + \frac{1}{6}h^3|c_k - y_k^{(5)}| + \frac{1}{24}h^4|y_k^{(6)} - y^{(6)}(n_k)|$$

$$\le \frac{5}{24} h^4 \omega_6(h),$$

$$\left|S_k^{(3)}(x) - y^{(3)}(x)\right| \le \frac{1}{2} h^3 \omega_6(h),$$

$$\left|S_k^{(4)}(x) - y^{(4)}(x)\right| \le h\left[\frac{h}{2}\omega_6(h)\right] + \frac{h^2}{2}[\omega_6(h)] = h^2 \omega_6(h),$$

$$\left|S_k^{(5)}(x) - y^{(5)}(x)\right| \le \frac{h}{2}\omega_6(h) + h\omega_6(h) = \frac{3}{2} h \omega_6(h).$$

The proof of theorem 3.2 complete.

Lemma 3.3: Let S(x) be the spline function in equation (1), and y(x) be a unique solution of any differential equations, satisfying $\lim_{m\to\infty} S^{(m)}(x_k) = \beta$ and $\lim_{m\to\infty} y^{(m)}(x_k) = \gamma$, then

$$\lim_{m\to\infty} \left|\mathrm{D}^{(m)}\mathrm{S}(\mathrm{x}_k) - \mathrm{D}^{(m)}\mathrm{y}(\mathrm{x}_k)\right| = 0.$$

Proof: From equation (1), the spline function $\mathrm{S}(\mathrm{x}_k)$ of finite dimension which is tend to zero, when take it the high derivatives, and $\mathrm{y}(\mathrm{x}_k)$ has a unique solution $y \in C^n[a,b]$, using (theorem 4.1, in Arvet Pedas [15]), we have

$$\max_{0\le k\le n} \left|\mathrm{D}^{(m)}\mathrm{S}(\mathrm{x}_k) - \mathrm{D}^{(m)}\mathrm{y}(\mathrm{x}_k)\right| = \left\|\mathrm{D}^{(m)}\mathrm{S}(\mathrm{x}_k) - \mathrm{D}^{(m)}\mathrm{y}(\mathrm{x}_k)\right\|_\infty \to 0$$

as $m \to \infty$, with knots, $\mathrm{x} \in [\mathrm{x}_k, \mathrm{x}_{k+1}], \mathrm{k} = 0, 1, 2, 3, \dots, \mathrm{n}-1$. Therefore, by taking Maximum for equation (11) in Theorem 3.2, we have for any step size h

$$\lim_{m\to\infty} \left|\mathrm{D}^{(m)}\mathrm{S}(\mathrm{x}_k) - \mathrm{D}^{(m)}\mathrm{y}(\mathrm{x}_k)\right| = 0.$$

Spline Algorithm and Computation

In this section, we describe of the orientation spline method algorithms based on the error estimations respect the theorem 3.2, which are obtained.

Step 1: Consider that the mentioned formulation and analysis was accomplished in $C^6[a,b]$.

Step 2: Using the Lemma 3.1, with initial values of x_0, y_0 to compute equation (1).

Step 3: Using step 2 to enumerate

$$S_k(x) = y_k + a_k(x - x_k) + \frac{1}{2}y_k''(x - x_k)^2 + \frac{1}{6}b_k(x - x_k)^3 + \frac{1}{24}y_k^{(4)}(x - x_k)^4 + \frac{1}{120}c_k(x - x_k)^5 + \frac{1}{720}y_k^{(6)}(x - x_k)^6,$$ in each subinterval $[\mathrm{x_k}, \mathrm{x_{k+1}}]$, $\mathrm{k} = 1, \dots, \mathrm{n}-1$

Step 4: The equations (8), (9) and (10), were used to find out error estimate between $\mathrm{S_k(x)}$ and y(x) in theorem 3.2, for each subinterval $[\mathrm{x_k}, \mathrm{x_{k+1}}]$, $\mathrm{k} = 1, \dots, \mathrm{n}-1$, where $i = 0,1,\dots,6$.

$$\left|S_k^{(i)}(x) - y^{(i)}(x)\right| \le c_i h^{6-i} \omega_6(h).$$

Numerical Computations

In this section, we perform numerical computations by Matlab programming using the sixtic spline interpolation with differences mesh size h and measurement bound error. We find the maximum absolute errors for two examples and plot the curve of these computations at difference values in the intervals.

Example 5.1: Consider the boundary value problems [9]

$$y^{(4)}(\mathrm{x}) = 6\exp\big(-4\mathrm{y}(\mathrm{x})\big) - 12(1+x)^{(-4)}, \qquad 0 < x < 1,$$

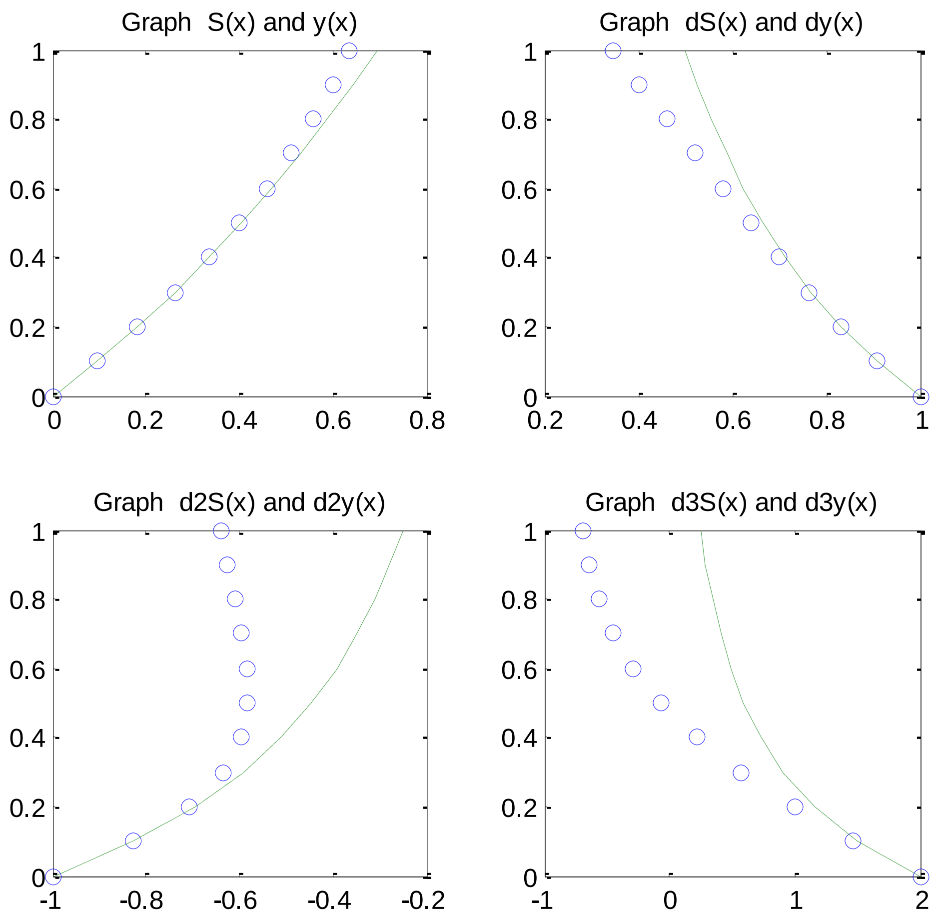
With the boundary conditions

$$y(0) = 0,\ \ \mathrm{y}(1) = \ln(2), \mathrm{y}^{(2)}(0) = 1, \qquad \mathrm{y}^{(2)}(1) = -0.25.$$

The analytic solution is $\mathrm{y}(x) = \ln(1+x)$.

Table 1, Maximum absolute errors for example 5.1

h	$\lvert y(t) - S(t)\rvert$	$\lvert y'(t) - S'(t)\rvert$	$\lvert y''(t) - S''(t)\rvert$	$\lvert y'''(t) - s'''(t)\rvert$
0.1	2.0697e-06	5.5507e-05	0.0014	0.0339
0.01	3.4542e-12	9.2634e-10	2.3551e-07	5.6523e-05
0.001	3.6447e-18	9.7745e-15	2.4851e-11	5.9641e-08



The error estimates between the approximate the solution 'o' and the exact solution '--' of example 5.1.

Example 5.2: Consider the following linear boundary value problem [16]

$$y^{(v)}(\mathrm{x}) = \mathrm{y} - 15e^{x} - 10xe^{x}$$

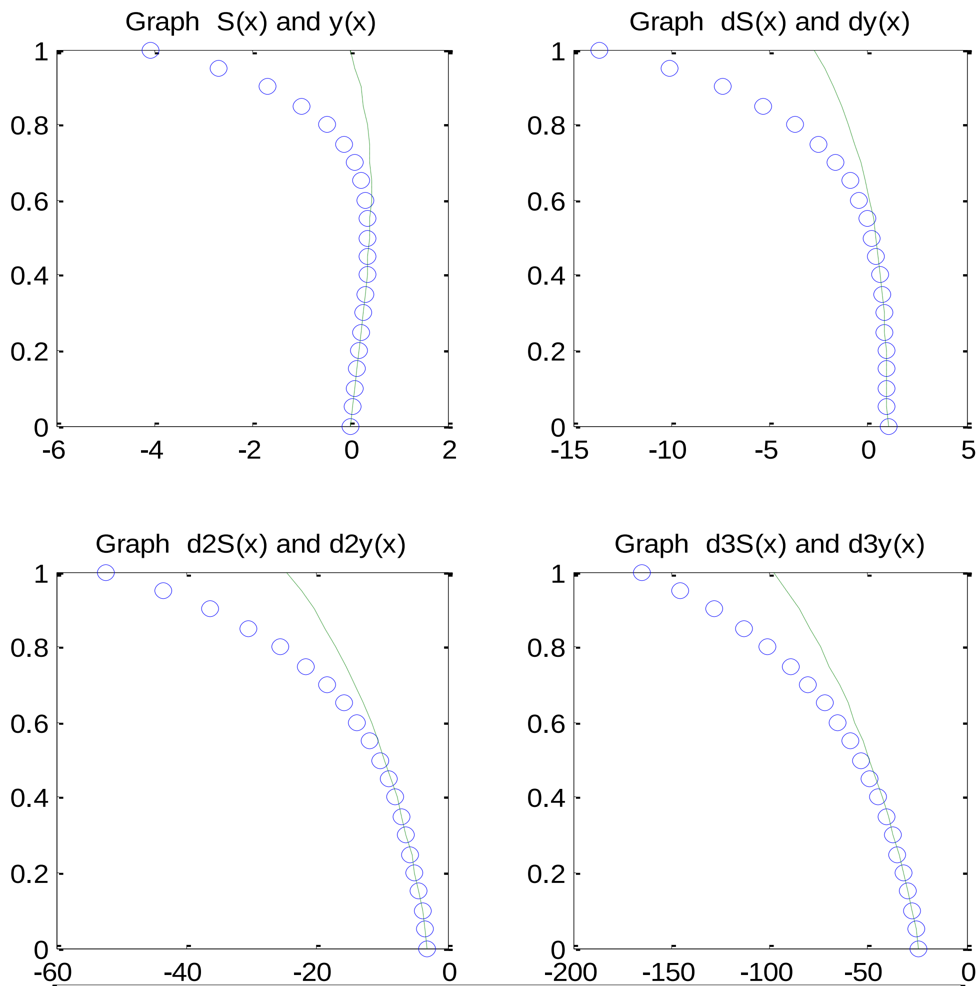
With the boundary conditions

$$y(0) = 0,\ \ \mathrm{y}'(1) = 1, \qquad \mathrm{y}''(0) = 0, \qquad \mathrm{y}(1) = 0, y'(1) = -e.$$

The exact solution is $\mathrm{y}(x) = \mathrm{x}(1 - x)e^{x}$.

Table 2, Maximum absolute errors for example 5.2

h	$\lvert \boldsymbol{y}(\boldsymbol{t}) - \boldsymbol{S}(\boldsymbol{t})\rvert$	$\lvert \boldsymbol{y}'(\boldsymbol{t}) - \boldsymbol{S}'(\boldsymbol{t})\rvert$	$\lvert \boldsymbol{y}''(\boldsymbol{t}) - \boldsymbol{S}''(\boldsymbol{t})\rvert$	$\lvert \boldsymbol{y}'''(\boldsymbol{t}) - \boldsymbol{s}'''(\boldsymbol{t})\rvert$
0.1	1.2262e-06	3.2883e-05	8.3602e-04	0.0201
0.01	1.0842e-12	2.9076e-10	7.3923e-08	1.7742e-05
0.001	1.0709e-18	2.8720e-15	7.3017e-12	1.7524e-08



The error estimates between the approximate the solution 'o' and the exact solution '--' of example 5.2.

Conclusions

In this paper, we applied sixtic degree spline polynomial algorithm to approximate the value of function in the numerical solution of differential equations can reduce the computational effort required to solve such problems because ability to use a mesh size of *h*. the numerical results, illustrated in Table 5.1 and 5.2, obvious that the sixtic spline interpolation and analytic solutions of the mesh points of interest can be readily plotted as the above figures.

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