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Estimations of the Upper Bound for the Eigen-Functions of the Fourth Order Boundary Value Problem with Smooth Coefficients

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Abstract: In this paper we consider a fourth-order boundary value problem with smooth coefficients. We found new expansions of linearly independent solutions that satisfy an initial condition. Then, by using these linearly independent solutions we found expressions for the eigenfunctions, and we also found an upper bound for the eigenfunctions and their derivatives.

Keywords: eigenvalue problem, spectral parameter, initial condition, boundary condition, fundamental system, eigenfunction, upper bound.

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1 Introduction

It is well-known that many topics in mathematical physics require the investigation of eigenvalues and eigenfunctions of Sturm Liouville type boundary value problems. In recent years, many researchers are interested in the continuous Sturm Liouville problem as we see in G. Hikmet, N. B. Kerimov, and U. Kaya in the article [1]. Also, H. Menken in the article [2] considered a nonself-adjoint fourth-order differential operator with periodic and anti-periodic boundary conditions. They found asymptotic formulas for the eigenvalues and eigenfunctions. Many authors investigated the method of determining a bound for eigenfunctions of a boundary value problem and its derivatives. As we see in [3,4] the authors found a bound for the eigenfunction and its derivatives of a spectral problem of the form -$y'' + q(x)y = \lambda^2\rho(x)y$. Various physics applications of this kind of a problem are found in the literature, including some boundary value problems with transmission conditions that arise in the theory of heat and mass transfer (see [5,6]). The literature on such results is voluminous, and we refer to [7]. Fourth-order continuous boundary value problems with eigenfunctions dependent boundary conditions and with two supplementary transmission conditions at the point of continuity have been investigated in [8]. We shall consider the fourth-order differential equation:

$$l(y) = y^{(4)}(x) + q(x)y(x) = \lambda^4 y(x), x \in [0,a] \tag{1}$$

$$U_j(y) = \begin{cases} y^{(j)}(0) = 0 & j = 0,1 \\ \sum_{i=1}^{4}(iw_j)^{i-1}y^{(4-i)}(a,\lambda) = 0 & j = 2,3 \end{cases} \tag{2}$$

This paper tries to estimate a new expression for the four linearly independent solutions of the differential equation (1) as well as their derivatives. In section 2, we found an expression for the eigenfunction of the boundary value problem (1)-(2), which is a linear combination of the linearly independent solutions. Also, in section 3-4, we obtain upper bounds for the eigenfunction and its derivatives that we obtained in section 2.

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2 Fundamental System of Solutions of the Differential equation

In this section, we find the expressions of four linearly independent solutions and their derivatives which satisfy the initial condition (4). These solutions mentioned in [9] for higher order. In ([7], pp. 92) and ([8], pp.5) two linearly independent solutions are found for the second order boundary value problem which is generated by the differential operator $l(y) = y''(x) + q(x)y(x)$ and they are used in a view to providing the existence of the eigenvalue for the problem $l(y) = \lambda^2 y(x)$. Here, we use the same technique to find four linearly independent solutions for the fourth-order differential equation (1). Also, we use these solutions to find a bound for the eigenfunctions of (1)-(2).

Theorem 2.1. Consider the linear differential equation of fourth order

$$y^{(4)}(x) + q(x)y(x) = \lambda^4 y(x) \tag{3}$$

Where $q(x)$ is a smooth function on $[0,a]$, then (3) has the fundamental system of solutions, $y_0(x,\lambda), y_1(x,\lambda), y_2(x,\lambda), y_3(x,\lambda)$, that satisfy the initial condition

$$y_i^{(n)}(0,\lambda) = \begin{cases} 1 & ,i = n-1 \\ 0 & ,i \neq n-1 \end{cases} \tag{4}$$

Where,

$$\begin{aligned} y_0(x,\lambda) &= \frac{1}{2}[\cosh\lambda x + \cos\lambda x] + \frac{1}{2\lambda^3}\int_0^x \Big[\sin\lambda(x-\xi) \\ &\quad + \sinh\lambda(x-\xi)\Big] q(\xi) y_0(x,\lambda) d\xi. \end{aligned} \tag{5}$$

$$\begin{aligned} y_1(x,\lambda) &= \frac{1}{2\lambda}[\sinh\lambda x + \sin\lambda x] + \frac{1}{\lambda^3}\int_0^x \Big[\sin\lambda(x-\xi) \\ &\quad + \sinh\lambda(x-\xi)\Big] q(\xi) y_1(x,\lambda) d\xi. \end{aligned} \tag{6}$$

$$\begin{aligned} y_2(x,\lambda) &= \frac{1}{2\lambda^2}[\cosh\lambda x - \cos\lambda x] + \frac{1}{\lambda^3}\int_0^x \Big[\sin\lambda(x-\xi) \\ &\quad + \sinh\lambda(x-\xi)\Big] q(\xi) y_2(x,\lambda) d\xi. \end{aligned} \tag{7}$$

$$\begin{aligned} y_3(x,\lambda) &= \frac{1}{2\lambda^3}[\sinh\lambda x - \sin\lambda x] + \frac{1}{2\lambda^3}\int_0^x \Big[\sin\lambda(x-\xi) \\ &\quad + \sinh\lambda(x-\xi)\Big] q(\xi) y_3(x,\lambda) d\xi. \end{aligned} \tag{8}$$

Proof. Consider the linear differential operator

$$l(y) = y^{(4)}(x) + q(x)y(x) \tag{9}$$

We want to find a non-zero solutions for

$$l(y) - \lambda^4 y = 0 \tag{10}$$

that satisfy the initial condition (4). First we reduce (10) to an integro-differential equation of the form

$$y^{(4)} - \lambda^4 y = m(y), m(y) = -q(x)y. \tag{11}$$

The homogeneous linear differential equation $y^{(4)} - \lambda^4 y = 0$ has for $\lambda \neq 0$ the solutions $e^{\lambda w_0 x}, e^{\lambda w_1 x}, e^{\lambda w_2 x}, e^{\lambda w_3 x}$. where $w_0 = 1, w_1 = i, w_2 = -1$, $w_3 = -i$. Then by using the method variation of parameters we can express the solutions of (11) for $k = 0,1,2,3$ as

$$y_k(x,\lambda) = c_0 e^{\lambda x} + c_1 e^{i\lambda x} + c_2 e^{-\lambda x} + c_3 e^{-i\lambda x} + \int_0^x \Big[\sin\lambda(x-\xi) + \sinh\lambda(x-\xi)\Big] q(\xi) y_3(x,\lambda) d\xi. \tag{12}$$

Now, applying (12) in (4), gives:

$$y_0(0,\lambda) = 1, y_0'(0,\lambda) = 0, y_0''(0,\lambda) = 0, y_0''(0,\lambda) = 0$$

Which gives a system of the form:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{13}$$

Solving it for c_j, we obtain $c_j = 1/4$ for each $j = 0:3$, then y_0 has the form

$$\begin{aligned} y_0(x,\lambda) &= \frac{1}{2}[\cosh\lambda x + \cos\lambda x] + \frac{1}{2\lambda^3}\int_0^x \Big[\sin\lambda(x-\xi) \\ &\quad + \sinh\lambda(x-\xi)\Big] q(\xi) y_0(x,\lambda) d\xi. \end{aligned} \tag{14}$$

Hence (5) is hold, and we can use the same technique for $y_1(x,\lambda), y_2(x,\lambda), y_3(x,\lambda)$ and we get (6), (7), (8).

Lemma 2.1. The first derivatives of the solutions of the differential equation (3) can be represented as the following expressions

$$\begin{aligned} y_0'(x,\lambda) &= \frac{1}{2}\lambda[\sinh\lambda x - \sin\lambda x] + \frac{1}{2\lambda^2}\int_0^x \Big[\cos\lambda(x-\xi) \\ &\quad + \cosh\lambda(x-\xi)\Big] q(\xi) y_0(x,\lambda) d\xi. \end{aligned} \tag{15}$$

$$\begin{aligned} y_1'(x,\lambda) &= \frac{1}{2}[\cosh\lambda x + \cos\lambda x] + \frac{1}{2\lambda^2}\int_0^x \Big[\cos\lambda(x-\xi) \\ &\quad + \cosh\lambda(x-\xi)\Big] q(\xi) y_1(x,\lambda) d\xi. \end{aligned} \tag{16}$$

$$\begin{aligned} y_2'(x,\lambda) &= \frac{1}{2\lambda}[\sinh\lambda x + \sin\lambda x] + \frac{1}{2\lambda^2}\int_0^x \Big[\cos\lambda(x-\xi) \\ &\quad + \cosh\lambda(x-\xi)\Big] q(\xi) y_2(x,\lambda) d\xi. \end{aligned} \tag{17}$$

$$y_3'(x,\lambda) = \frac{1}{2\lambda^2}[\cosh\lambda x - \cos\lambda x] + \frac{1}{2\lambda^2}\int_0^x \Big[\cos\lambda(x-\xi) + \cosh\lambda(x-\xi)\Big] q(\xi) y_2(\xi,\lambda) d\xi. \tag{18}$$

Lemma 2.2. The second derivatives of the solutions of the differential equation (3) can be represented as the following expressions

$$y_0''(x,\lambda) = \frac{1}{2}\lambda^2[\cosh\lambda x - \cos\lambda x] + \frac{1}{2\lambda}\int_0^x \Big[-\sin\lambda(x-\xi) + \sinh\lambda(x-\xi)\Big] q(\xi) y_0(\xi,\lambda) d\xi. \tag{19}$$

$$y_1''(x,\lambda) = \frac{1}{2}\lambda[\sinh\lambda x - \sin\lambda x] + \frac{1}{2\lambda}\int_0^x \Big[-\sin\lambda(x-\xi) + \sinh\lambda(x-\xi)\Big] q(\xi) y_0(\xi,\lambda) d\xi. \tag{20}$$

$$y_2''(x,\lambda) = \frac{1}{2}[\cosh\lambda x + \cos\lambda x] + \frac{1}{2\lambda}\int_0^x \Big[-\sin\lambda(x-\xi) + \sinh\lambda(x-\xi)\Big] q(\xi) y_0(\xi,\lambda) d\xi. \tag{21}$$

$$y_3''(x,\lambda) = \frac{1}{2\lambda}[\sinh\lambda x + \sin\lambda x] + \frac{1}{2\lambda}\int_0^x \Big[-\sin\lambda(x-\xi) + \sinh\lambda(x-\xi)\Big] q(\xi) y_0(\xi,\lambda) d\xi. \tag{22}$$

Lemma 2.3. The third derivatives of the solutions of the differential equation (3) can be represented as the following expressions

$$y_0'''(x,\lambda) = \frac{1}{2}\lambda^3[\sinh\lambda x + \sin\lambda x] + \frac{1}{2}\int_0^x \Big[-\cos\lambda(x-\xi) + \cosh\lambda(x-\xi)\Big] q(\xi) y_0(\xi,\lambda) d\xi. \tag{23}$$

$$y_1'''(x,\lambda) = \frac{1}{2}\lambda^2[\cosh\lambda x - \cos\lambda x] + \frac{1}{2}\int_0^x \Big[-\cos\lambda(x-\xi) + \cosh\lambda(x-\xi)\Big] q(\xi) y_0(\xi,\lambda) d\xi. \tag{24}$$

$$y_2'''(x,\lambda) = \frac{1}{2}\lambda[\sinh\lambda x - \sin\lambda x] + \frac{1}{2}\int_0^x \Big[-\cos\lambda(x-\xi) + \cosh\lambda(x-\xi)\Big] q(\xi) y_0(\xi,\lambda) d\xi. \tag{25}$$

$$y_3'''(x,\lambda) = \frac{1}{2}[\cosh\lambda x + \cos\lambda x] + \frac{1}{2}\int_0^x \Big[-\cos\lambda(x-\xi) + \cosh\lambda(x-\xi)\Big] q(\xi) y_0(\xi,\lambda) d\xi. \tag{26}$$

3 Estimations for bounds of eigenfunctions of boundary value problem (1)-(2)

In this section, we found an upper bound for the eigenfunctions of the problem (1)-(2), which can be represented as a linear combination of the solutions (5)-(8). But first, we studied the simplicity of the eigenfunctions of the boundary value problem (1)-(2).

Theorem 3.1 The eigenfunctions of the boundary value problem (1)-(2) are simple.

Proof. let λ be an eigenvalue for the boundary value problem (1)-(2). On the contrary, we suppose that $\phi_1$and ϕ_2 are two linearly independent eigenfunctions corresponding to λ, then $\phi_1$and ϕ_2 satisfy the differential equation and the boundary conditions. Now, from first and second boundary conditions we get

$$\phi_1(0,\lambda) + \phi_1'(0,\lambda) = 0 \tag{27}$$

$$\phi_2(0,\lambda) + \phi_2'(0,\lambda) = 0 \tag{28}$$

which can be represented as the system:

$$\begin{bmatrix} \phi_1(0,\lambda) & \phi_1'(0,\lambda) \\ \phi_2(0,\lambda) & \phi_2'(0,\lambda) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{29}$$

Since $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so system (29) has non-zero solutions, and from the theory of systems we found that $\phi_1(0,\lambda)\phi_2'(0,\lambda) - \phi_1'(0,\lambda)\phi_2(0,\lambda) = 0$. Again, from the theory of systems of differential equations, we get that $\phi_1(x,\lambda)$ and $\phi_2(x,\lambda)$ are linearly dependent at $x = 0$. Then, $\phi_2(x,\lambda)$ are linearly dependent for every x, which contradicts our assumption.

Theorem 3.2. If $\lambda = r + it$ is an eigenvalue of boundary value problem (1)-(2) and $\phi_n(x,\lambda)$ is the eigen-function of the boundary value problem(1)-(2) corresponding to λ , and

$$\int_0^a |q(\xi)||\phi(\xi,\lambda)| d\xi < \infty$$

Then,

$$max_{x\in[0,a]}|\phi_n(x,\lambda)| \leq 2Ce^{\int_0^a |q(\xi)||\phi(\xi,\lambda)|d\xi} \tag{30}$$

where,

$$C = \begin{cases} \frac{M^{3|t|}}{|\lambda|^3} & ,|r| \leq |t| \\ \frac{M^{3|r|}}{|\lambda|^3} & ,|r| \geq |t| \end{cases} \tag{31}$$

Proof. If $\phi_n(x)$ is an eigen-function of the differential equation, which implies that it is a solution for the differential equation, then it can be written as a linear combination of y_0, y_1, y_2, y_3. That is, there exists constants a_0, a_1, a_2, a_3 such that:

$$\phi_n(x,\lambda) = a_0 y_0(x,\lambda) + a_1 y_1(x,\lambda) + a_2 y_2(x,\lambda) + a_3 y_3(x,\lambda) \tag{32}$$

where $y_0(x,\lambda), y_1(x,\lambda), y_2(x,\lambda)$ and $y_3(x,\lambda)$ are the fundamental system of solutions in the Theorem (2). Also, ϕ_n satisfies the boundary condition (2). From first boundary condition, we get

$$\begin{aligned}\phi_n(0,\lambda) = a_0y_0(0,\lambda) &+a_1y_1(0,\lambda)+a_2y_2(0,\lambda)\\ &+a_3y_3(0,\lambda)=0.\end{aligned}$$

and from second boundary condition, we get

$$\begin{aligned}\phi_n^{'}(0,\lambda) = a_0y_0^{'}(0,\lambda) &+a_1y_1^{'}(0,\lambda)+a_2y_2^{'}(0,\lambda)\\ &+a_3y_3^{'}(0,\lambda)=0.\end{aligned}$$

Hence $a_0 = 0$ and $a_1 = 0$. Thus, the eigenfunction and its derivatives reduce to the following expressions

$$\begin{aligned}\phi_n(x,\lambda) &= a_2\frac{1}{2\lambda^2}[\cosh\lambda x-\cos\lambda x]+a_3\frac{1}{2\lambda^3}[\sinh\lambda x-\\ &\sin\lambda x]+\frac{1}{2\lambda^3}\int_0^x\Big[\sin\lambda(x-\xi)+\sinh\lambda(x-\xi)\\ &\Big]q(\xi)\phi_n(\xi,\lambda)d\xi \qquad (33)\end{aligned}$$

$$\begin{aligned}\phi_n^{'}(x,\lambda) &= a_2\frac{1}{2\lambda}[\sinh\lambda x+\sin\lambda x]+a_3\frac{1}{2\lambda^2}[\cosh\lambda x-\\ &\cos\lambda x]+\frac{1}{2\lambda^2}\int_0^x\Big[\cos\lambda(x-\xi)+\cosh\lambda(x-\xi)\\ &\Big]q(\xi)\phi_n(\xi,\lambda)d\xi \qquad (34)\end{aligned}$$

$$\begin{aligned}\phi_n^{''}(x,\lambda) &= a_2\frac{1}{2}[\cosh\lambda x+\cos\lambda x]+a_3\frac{1}{2\lambda}[\sinh\lambda x+\\ &\sin\lambda x]+\frac{1}{2\lambda}\int_0^x\Big[-\sin\lambda(x-\xi)+\sinh\lambda(x-\xi)\\ &\Big]q(\xi)\phi_n(\xi,\lambda)d\xi \qquad (35)\end{aligned}$$

$$\begin{aligned}\phi_n^{'''}(x,\lambda) &= a_2\frac{1}{2}\lambda[\sinh\lambda x-\sin\lambda x]\\ &+a_3\frac{1}{2}[\cosh\lambda x+\cos\lambda x]\\ &+\frac{1}{2\lambda}\int_0^x\Big[-\cos\lambda(x-\xi)+\cosh\lambda(x-\xi)\\ &\Big]q(\xi)\phi_n(\xi,\lambda)d\xi \qquad (36)\end{aligned}$$

Also, from second boundary condition, we obtain:

$$\begin{aligned}U_2(\phi_n(x)) &= \Bigg[a_2\Big[\lambda[-\sin\lambda a]+i\lambda[-\cos\lambda a]\Big]\\ &+a_3\Big[[cos\lambda a]+i[-\sin\lambda a]\Big]\\ &+\int_0^a\Big[isin\lambda(a-\xi)-\cos\lambda(a-\xi)\\ &\Big]q(\xi)\phi_n(\xi,\lambda)d\xi\Bigg]=0 \qquad (37)\end{aligned}$$

Now, since e^x is continuous in $[0,a]$, so it has a maximum value in $[0,a]$. We put $max_{\xi\in[0,a]}e^{\xi}=M$. So $e^{\xi}\le M$, and since $|t|,|r|\ge 0$ then $e^{|t|\xi}\le M^{|t|}$ and $e^{|r|\xi}\le M^{|r|}$. And since $|t|,|r|\in R$, so we have $|t|\le|r|$ or $|r|\le|t|$, if $|r|\le|t|$ then $\xi|r|\le\xi|t|$ for all $\xi\in[0,a]$, then $e^{\xi|r|}\le e^{\xi|t|}\le M^{|t|}$. If

$$I=-\int_0^a[isin\lambda(a-\xi)-cos\lambda(a-\xi)]q(\xi)\phi_n(\xi,\lambda)d\xi \qquad (38)$$

,then

$$\begin{aligned}|I| &= |\int_0^a[i\sin\lambda(a-\xi)-\cos\lambda(a-\xi)]q(\xi)\phi_n(\xi,\lambda)d\xi|\\ &\le\int_0^a[|sin\lambda(a-\xi)|+|cos\lambda(a-\xi)|]|q(\xi)||\phi_n(\xi,\lambda)|d\xi\\ &\le\int_0^a[e^{|t|(a-\xi)}+e^{|t|(a-\xi)}]|q(\xi)||\phi_n(\xi,\lambda)|d\xi\\ &\le 2M^{|t|}\int_0^a|q(\xi)||\phi_n(\xi,\lambda)|d\xi.\end{aligned}$$

So

$$|I|\le 2M^{|t|}\int_0^a|q(\xi)||\phi_n(\xi,\lambda)|d\xi. \qquad (39)$$

And

$$G=[i[-\sin\lambda a]+[\cos\lambda a]] \qquad (40)$$

,then

$$\begin{aligned}|G| &= |-i\sin\lambda a+\cos\lambda a|\\ &= |-i\frac{e^{i\lambda a}-e^{-i\lambda a}}{2i}+\frac{e^{i\lambda a}+e^{-i\lambda a}}{2}|\\ &= \frac{1}{2}|-e^{i\lambda a}+e^{-i\lambda a}+e^{i\lambda a}+e^{-i\lambda a}|\\ &= \frac{1}{2}|e^{-i\lambda a}+e^{-i\lambda a}|\\ &= |e^{-i\lambda a}|\\ &= |e^{t-ir}a|\\ &= e^{ta}\ge e^{-|t|a}\ge M^{-|t|}.\end{aligned}$$

So

$$\frac{1}{|G|}\le M^{|t|}. \qquad (41)$$

Then, from (37) we obtain an expression for a_2 in terms of a_3 as follows

$$\begin{aligned}a_2\Big[\lambda[-\sin\lambda a] &+i\lambda[-\cos\lambda a]\Big]\\ &+a_3\Big[[\cos\lambda a]+i[-\sin\lambda a]]-I\Big]=0.\end{aligned}$$

$$a_2=\frac{I-a_3[[\cos\lambda a]+i[-\sin\lambda a]]}{-i\lambda[i[-\sin\lambda a]+[cos\lambda a]]}=\frac{a_3G-I}{i\lambda G}.$$

Thus,

$$a_2 = \frac{a_3 G - I}{i\lambda G}. \tag{42}$$

So, the eigenfunction of the boundary value problem (1)-(2) has the following form

$$\begin{aligned}\phi_n(x,\lambda) \;&= \Bigg[\frac{G-I}{i\lambda G}\frac{1}{2\lambda^2}[\cosh\lambda x - \cos\lambda x] + \frac{1}{2\lambda^3}[\sinh\lambda x \\ &- \sin\lambda x] + \frac{1}{2\lambda^3}\int_0^x \big[\sin\lambda(x-\xi) + \sinh\lambda(x-\xi) \\ &\big] q(\xi)\phi_n(\xi,\lambda)d\xi\Bigg]. \end{aligned}\tag{43}$$

Now, we want to find the maximum of $|\phi_n(x,\lambda)|$. We have

$$\begin{aligned}|\phi(x,\lambda)| \;&= \Bigg|[i(-1+\frac{I}{G})\frac{1}{2\lambda^3}[\cosh\lambda x - cos\lambda x] \\ &+\frac{1}{2\lambda^3}[\sinh\lambda x - \sin\lambda x] \\ &+\frac{1}{2\lambda^3}\int_0^x [\sin\lambda(x-\xi) \\ &+\sinh\lambda(x-\xi)]q(\xi)\phi_n(\xi,\lambda)d\xi]\Bigg|. \\ &\le |i(-1+\frac{I}{G})|\frac{1}{2|\lambda|^3}[|\cosh\lambda x| + |\cos\lambda x|] \\ &+\frac{1}{2|\lambda|^3}[|\sinh\lambda x| + |\sin\lambda x|] \\ &+\frac{1}{2|\lambda|^3}\int_0^x [|sin\lambda(x-\xi)| + |sinh\lambda(x-\xi)| \\ &]|q(\xi)||\phi_n(\xi,\lambda)|d\xi \end{aligned}$$

That means,

$$\begin{aligned}|\phi(x,\lambda)| \;&\le \frac{1}{2|\lambda|^3}\Bigg[|i(-1+\frac{I}{G})|[|\cosh\lambda x| + |\cos\lambda x|] \\ &+[|\sinh\lambda x| + |\sin\lambda x|] + \int_0^x [|\sin\lambda(x-\xi)| \\ &+|\sinh\lambda(x-\xi)|]|q(\xi)||\phi_n(\xi,\lambda)|d\xi\Bigg] \end{aligned}\tag{44}$$

If $\lambda = r+it$, and by using the following relations

$$\begin{aligned}&|\cos z| \le e^{|Imz|}, &&|\sin z| \le e^{|Imz|}, \\ &|\cosh z| \le e^{|Rez|}, &&|\sinh z| \le e^{|Rez|}. \end{aligned}\tag{45}$$

Then, by replacing $z = \lambda x$ in the above relations we get:

$$\begin{aligned}&|\cos\lambda x| \le e^{|t|x}, &&|\sin\lambda x| \le e^{|t|x}, \\ &|\cosh\lambda x| \le e^{|r|x}, &&|\sinh\lambda x| \le e^{|r|x}. \end{aligned}\tag{46}$$

and

$$\begin{aligned}&|\cos\lambda(x-\xi)| \le e^{|t|(x-\xi)}, |\cosh\lambda(x-\xi)| \le e^{|r|(x-\xi)}, \\ &|\sin\lambda(x-\xi)| \le e^{|t|(x-\xi)}, |\sinh\lambda(x-\xi)| \le e^{|r|(x-\xi)}. \end{aligned}\tag{47}$$

Therefore, (44) has the following form:

$$\begin{aligned}|\phi_n(x,\lambda)| \;&\le \frac{M^{|t|}}{|\lambda|^3}[2+\frac{|I|}{|G|}] \\ &+\frac{M^{|t|}}{|\lambda|^3}\int_0^a |q(\xi)||\phi_n(\xi,\lambda)|d\xi. \end{aligned}\tag{48}$$

Again, from (39) and (41), we get :

$$|\phi_n(x,\lambda)| \le 2Ce^{3C\int_0^a |q(\xi)||\phi_n(\xi,\lambda)|d\xi}. \tag{49}$$

Where, $C = \frac{M^{3|t|}}{|\lambda|^3}$. And since we choose x as any arbitrary number in $[0,a]$, then

$$max_{x\in[0,a]}|\phi_n(x,\lambda)| \le 2Ce^{3C\int_0^a |q(\xi)||\phi_n(\xi,\lambda)|d\xi}. \tag{50}$$

4 Estimations for bounds of the derivatives of the eigenfunctions of boundary value problem (1)-(2)

This section studies the method of finding a bound for the derivatives of eigenfunctions for the problem (1)-(2).

Theorem 3.1. If $\lambda = r+it$ is an eigenvalue of boundary value problem (1)-(2) and $\phi_n(x,\lambda)$ is the eigen-function of the boundary value problem(1)-(2) corresponding to λ , and

$$\int_0^a |q(\xi)||\phi(\xi,\lambda)|d\xi < \infty$$

Then,

$$max_{x\in[0,a]}|\phi_n^{(j)}(x,\lambda)| \le \begin{cases}\frac{M^{3|t|}}{|\lambda|^{3-j}}C & ,|r| \le |t| \\ \frac{M^{3|r|}}{|\lambda|^{3-j}}C & ,|r| \ge |t|\end{cases}. \tag{51}$$

for $j = 1,2,3$. where, $C = \int_0^a |q(\xi)||\phi(\xi,\lambda)|d\xi$.

Proof. Since from (43) we have

$$\begin{aligned}\phi_n(x,\lambda) \;&= \Bigg[\frac{G-I}{i\lambda G}\frac{1}{2\lambda^2}[\cosh\lambda x - \cos\lambda x] + \frac{1}{2\lambda^3}[\sinh\lambda x \\ &- \sin\lambda x] + \frac{1}{2\lambda^3}\int_0^x \big[\sin\lambda(x-\xi) + \sinh\lambda(x-\xi) \\ &\big] q(\xi)\phi_n(\xi,\lambda)d\xi\Bigg]. \end{aligned}\tag{52}$$

Then we can find the first, second and third derivatives of $\phi_n(x,\lambda)$ in $[0,a]$, which they have the following expressions:

$$\begin{aligned}\phi_n^{'}(x,\lambda) \;&= \Bigg[\frac{G-I}{i\lambda G}\frac{1}{2\lambda}[\sinh\lambda x + \sin\lambda x] + \frac{1}{2\lambda^2}[\cosh\lambda x \\ &- \cos\lambda x] + \frac{1}{2\lambda^2}\int_0^x \big[\cos\lambda(x-\xi) \\ &+\cosh\lambda(x-\xi)\big] q(\xi)\phi_n(\xi,\lambda)d\xi\Bigg]. \end{aligned}\tag{53}$$

$$\begin{aligned}\phi_n^{''}(x,\lambda) &= \Bigg[\frac{G-I}{i\lambda G}\frac{1}{2}[\cosh\lambda x+\cos\lambda x]+\frac{1}{2\lambda}[\sinh\lambda x\\ &+\sin\lambda x]+\frac{1}{2\lambda}\int_0^x\big[-\sin\lambda(x-\xi)\\ &+\sinh\lambda(x-\xi)\big]q(\xi)\phi_n(\xi,\lambda)d\xi\Bigg].\end{aligned}\tag{54}$$

$$\begin{aligned}\phi_n^{'''}(x,\lambda) &= \Bigg[\frac{G-I}{iG}\frac{1}{2}[\sinh\lambda x-\sin\lambda x]+\frac{1}{2}[\cosh\lambda x\\ &+\cos\lambda x]+\frac{1}{2}\int_0^x\big[-\cos\lambda(x-\xi)\\ &+\cosh\lambda(x-\xi)\big]q(\xi)\phi_n(\xi,\lambda)d\xi\Bigg].\end{aligned}\tag{55}$$

$$\begin{aligned}|\phi_n^{'}(x,\lambda)| &= \Bigg|\Bigg[\frac{G-I}{i\lambda G}\frac{1}{2\lambda}[\sinh\lambda x+\sin\lambda x]+\\ &\frac{1}{2\lambda^2}[\cosh\lambda x-\cos\lambda x]\\ &+\frac{1}{2\lambda^2}\int_0^x\big[\cos\lambda(x-\xi)+\cosh\lambda(x-\xi)\\ &\big]q(\xi)\phi_n(\xi,\lambda)d\xi\Bigg]\Bigg|.\\ &\le \Bigg[|\frac{G-I}{i\lambda G}|\frac{1}{2|\lambda|}[|\sinh\lambda x|+|\sin\lambda x|]\\ &+\frac{1}{2|\lambda|^2}[|\cosh\lambda x|-|\cos\lambda x|]\\ &+\frac{1}{2|\lambda|^2}\int_0^x\big[|\cos\lambda(x-\xi)|+|\cosh\lambda(x-\xi)|\\ &\big]|q(\xi)||\phi_n(\xi,\lambda)|d\xi\Bigg].\\ &\le |-1+\frac{I}{G}|\frac{1}{|\lambda|^2}|M^{|t|}+\frac{1}{|\lambda|^2}M^{|t|}\\ &+\frac{M^{|t|}}{|\lambda|^2}\int_0^x|q(\xi)||\phi_n(\xi,\lambda)|d\xi.\end{aligned}\tag{56}$$

Simple calculation of the above inequality gives the following inequality

$$\begin{aligned}|\phi_n^{'}(x,\lambda)| \le\ &\frac{M^{|t|}}{|\lambda|^2}[2+|\frac{I}{G}|]\\ &+\frac{M^{|t|}}{|\lambda|^2}\int_0^x|q(\xi)||\phi_n(\xi,\lambda)|d\xi.\end{aligned}\tag{57}$$

By using (39) and (41), the right-hand side of the above inequality reduces to

$$\begin{aligned}&[2\frac{M^{|t|}}{|\lambda|^2}\ +2\frac{M^{3|t|}}{|\lambda|^2}\int_0^x|q(\xi)||\phi_n(\xi,\lambda)|d\xi]\\ &\quad+\frac{M^{|t|}}{|\lambda|^2}\int_0^x|q(\xi)||\phi_n(\xi,\lambda)|d\xi.\end{aligned}\tag{58}$$

Finally, we obtain

$$\begin{aligned}|\phi_n^{'}(x,\lambda)| \le [2\frac{M^{|t|}}{|\lambda|^2}\ &+2\frac{M^{3|t|}}{|\lambda|^2}\int_0^x|q(\xi)||\phi_n(\xi,\lambda)|d\xi]\\ &+\frac{M^{|t|}}{|\lambda|^2}\int_0^x|q(\xi)||\phi_n(\xi,\lambda)|d\xi.\end{aligned}\tag{59}$$

By calculating the right-hand side of this inequality as follows

$$\begin{aligned}|\phi_n^{'}(x,\lambda)| &\le [2\frac{M^{|t|}}{|\lambda|^2}\\ &+(2\frac{M^{3|t|}}{|\lambda|^2}+\frac{M^{|t|}}{|\lambda|^2})\int_0^x|q(\xi)||\phi_n(\xi,\lambda)|d\xi].\\ &\le [2\frac{M^{3|t|}}{|\lambda|^2}\\ &+(3\frac{M^{3|t|}}{|\lambda|^2})\int_0^x|q(\xi)||\phi_n(\xi,\lambda)|d\xi].\\ &\le \frac{M^{3|t|}}{|\lambda|^2}[2+3\int_0^x|q(\xi)||\phi_n(\xi,\lambda)|d\xi]\\ &\le \frac{M^{3|t|}}{|\lambda|^2}C\end{aligned}$$

where, $C=[2+3\int_0^x|q(\xi)||\phi_n(\xi,\lambda)|d\xi]$.
This means that:

$$|\phi_n^{'}(x,\lambda)| \ \le \frac{M^{3|t|}}{|\lambda|^2}C$$

Since x is arbitrary in $[0,a]$ then we can say that

$$max_{x\in[0,a]}|\phi_n^{'}(x,\lambda)|\le\frac{M^{3|t|}}{|\lambda|^2}C.\tag{60}$$

From (53), we have

$$
\begin{aligned}
|\phi_n''(x,\lambda)| &= \left| \frac{G-I}{i\lambda G}\frac{1}{2}[\cosh\lambda x + \cos\lambda x] + \frac{1}{2\lambda}[\sinh\lambda x \right. \\
&\quad + \sin\lambda x] + \frac{1}{2\lambda}\int_0^x \big[-\sin\lambda(x-\xi) \\
&\quad \left. + \sinh\lambda(x-\xi)\big]q(\xi)\phi_n(\xi,\lambda)d\xi \right| \\
&\leq |-1+\frac{I}{G}|\frac{1}{2|\lambda|}[|\cosh\lambda x| + |\cos\lambda x|] \\
&\quad + \frac{1}{2|\lambda|}[|\sinh\lambda x| + |\sin\lambda x|] \\
&\quad + \frac{1}{2|\lambda|}\int_0^x \big[|\sin\lambda(x-\xi)| + |\sinh\lambda(x-\xi)| \\
&\quad \left. \big]|q(\xi)||\phi_n(\xi,\lambda)|d\xi \right| \qquad (61)
\end{aligned}
$$

Simple calculation leads to

$$
\begin{aligned}
|\phi_n''(x,\lambda)| &\leq \frac{M^{|t|}}{|\lambda|}[2+|\frac{I}{G}|] + \frac{M^{|t|}}{|\lambda|}\int_0^x |q(\xi)||\phi_n(\xi,\lambda)|d\xi \\
&\leq [2\frac{M^{|t|}}{|\lambda|} + 2\frac{M^{3|t|}}{|\lambda|}\int_0^x |q(\xi)||\phi_n(\xi,\lambda)|d\xi \\
&\quad + \frac{M^{|t|}}{|\lambda|}\int_0^x |q(\xi)||\phi_n(\xi,\lambda)|d\xi.
\end{aligned}
$$

By using the same techniques as we did for $|\phi_n'(x,\lambda)|$, we obtain

$$
\begin{aligned}
|\phi_n''(x,\lambda)| &\leq 2\frac{M^{|t|}}{|\lambda|} \\
&\quad + (2\frac{M^{3|t|}}{|\lambda|} + \frac{M^{|t|}}{|\lambda|})\int_0^x |q(\xi)||\phi_n(\xi,\lambda)|d\xi] \\
&\leq [2\frac{M^{3|t|}}{|\lambda|} + (3\frac{M^{3|t|}}{|\lambda|})\int_0^x |q(\xi)||\phi_n(\xi,\lambda)|d\xi] \\
&\leq \frac{M^{3|t|}}{|\lambda|}[2+3\int_0^x |q(\xi)||\phi_n(\xi,\lambda)|d\xi] \leq \frac{M^{3|t|}}{|\lambda|}C.
\end{aligned}
$$

Where $C = [2+3\int_0^x |q(\xi)||\phi_n(\xi,\lambda)|d\xi]$. Thus, we have

$$
|\phi_n''(x,\lambda)| \leq \frac{M^{3|t|}}{|\lambda|}C. \qquad (62)
$$

Again, x is arbitrary in $[0,a]$ so we obtain

$$
max_{x\in[0,a]}|\phi_n''(x,\lambda)| \leq \frac{M^{3|t|}}{|\lambda|}C. \qquad (63)
$$

Next, from (55) we have

$$
\begin{aligned}
|\phi_n'''(x,\lambda)| &= \left| \frac{G-I}{iG}\frac{1}{2}[\sinh\lambda x - \sin\lambda x] + \frac{1}{2}[\cosh\lambda x \right. \\
&\quad + \cos\lambda x] + \frac{1}{2}\int_0^x \big[-\cos\lambda(x-\xi) \\
&\quad \left. + \cosh\lambda(x-\xi)\big]q(\xi)\phi_n(\xi,\lambda)d\xi \right|. \\
&\leq |\frac{G-I}{iG}|\frac{1}{2}[|\sinh\lambda x| + |\sin\lambda x|] + \frac{1}{2}[|\cosh\lambda x| \\
&\quad + |\cos\lambda x|] + \frac{1}{2}\int_0^x \big[|\cos\lambda(x-\xi)| \\
&\quad + |\cosh\lambda(x-\xi)|\big]q(\xi)||\phi_n(\xi,\lambda)|d\xi. \qquad (64)
\end{aligned}
$$

Some simple calculations give:

$$
\begin{aligned}
|\phi_n'''(x,\lambda)| &\leq M^{|t|}[2+|\frac{I}{G}|] + M^{|t|}\int_0^x |q(\xi)||\phi_n(\xi,\lambda)|d\xi \\
&\leq [2M^{|t|} + 2M^{3|t|}\int_0^x |q(\xi)||\phi_n(\xi,\lambda)|d\xi] \\
&\quad + M^{|t|}\int_0^x |q(\xi)||\phi_n(\xi,\lambda)|d\xi.
\end{aligned}
$$

By using the same techniques as we did for $|\phi_n'(x,\lambda)|$, we obtain:

$$
\begin{aligned}
|\phi_n'''(x,\lambda)| &\leq \Big[2M^{|t|} \\
&\quad + (2M^{3|t|} + M^{|t|})\int_0^a |q(\xi)||\phi_n(\xi,\lambda)|d\xi\Big] \\
&\leq [2M^{3|t|} + (3M^{3|t|})\int_0^a |q(\xi)||\phi_n(\xi,\lambda)|d\xi] \\
&\leq M^{3|t|}[2+3\int_0^a |q(\xi)||\phi_n(\xi,\lambda)|d\xi \\
&\leq M^{3|t|}C,
\end{aligned}
$$

where $C = [2+3\int_0^a |q(\xi)||\phi_n(\xi,\lambda)|d\xi]$. That is,

$|\phi_n'''(x,\lambda)| \leq M^{3|t|}C,$

Again, x is arbitrary in $[0,a]$ so we obtain

$$
max_{x\in[0,a]}|\phi_n'''(x,\lambda)| \leq M^{3|t|}C. \qquad (65)
$$

So by the above calculations, we proved that the derivatives of the eigenfunctions of the boundary value problem (1)-(2) are bounded. For complex number $\lambda = r+it$ and $|r| \leq |t|$

$$
max_{x\in[0,a]}|\phi_n^{(j)}(x,\lambda)| \leq \frac{M^{3|t|}}{|\lambda|^{3-j}}C, j=1,2,3. \qquad (66)
$$

And for the complex number $\lambda = r+it$ and $|r| \geq |t|$ we repeat the same process as above, and we get

$$
max_{x\in[0,a]}|\phi_n^{(j)}(x,\lambda)| \leq \frac{M^{3|r|}}{|\lambda|^{3-j}}C, j=1,2,3. \qquad (67)
$$

Hence, Theorem 4 was proved.

5 Conclusion

In this work, we found four linearly independent solutions of the boundary value problem (1)-(2) and then we used these solutions to get the expressions of the eigenfunctions and their derivatives. We also proved that all the eigenfunctions of the boundary value problem (1)-(2) are simple, and finally, we obtained upper bounds for the eigenfunctions and their derivatives.

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