On SC-FDMA Resource Allocation with Power Control

Ghafour Ahani
Department of Computer Engineering
Komar University of Science and Technology
Sulaymaniyah, Iraq
Email: ghafour.ahani@komar.edu.iq

Di Yuan
Department of Science and Technology
Linköping University
Norrköping, Sweden
Email: di.yuan@liu.se

Wei Ding
Ranplan Wireless Network Design Ltd
Cambourne, Cambridge CB23 6DP, United Kingdom
Email: wei.ding@ranplan.co.uk

***Abstract*— In this paper, single-carrier frequency division multiple accesses (SC-FDMA) is discussed. In particular, minimum sum power, subject to meeting user's demand is considered. There are two channel allocation schemes, localized and interleaved. In localized scheme, a block of convective channels in the spectrum is allocated to each user. In interleaved scheme, channels will be spread out over the spectrum and will be distributed equidistantly. It has been previously assumed that when a block of channels is assigned to a user, the same amount of power will be allocated to each channel. However, the power could be used more efficiently without this assumption. We show that the resulting power allocation problem can be solved in linear time and propose an optimal power allocation procedure. Next, the effect of this new power optimization procedure is investigated numerically. In the next part of paper, we prove that for the interleaved scheme, Minimum sum power problem with or without this new power optimization is polynomial solvable. Finally, we numerically compare localized and interleaved SC-FDMA with and without power optimization. The results show that the localized scheme with the new power optimization yields the best performance**

Keywords*— *algorithm; channel allocation; optimization; single carrier frequency division multiple access

I. Introduction

Orthogonal frequency division multiplexing accesses (OFDMA) has been a prominent technique for high data rate downlink communication in 4G systems [1]. An important advantage of OFDMA is its robustness in the presence of multi-path fading in cellular applications [2]. However, its main disadvantage in uplink communication is high peak-to-average ratio (PAPR) [1]. SC-FDMA, a modified version of OFDMA, has become very popular as it has shown to propose a lower PAPR compared to OFDMA [1]. Consequently, uplink communication with SC-FDMA consumes less energy, so data can be transmitted at a smaller backoff from the peak power. SC-FDMA has been adopted as the uplink multiple access scheme in the Third Generation Partnership Project Long Term Evolution (3GPP-LTE) standard [3].

Localized and interleaved channel assignments are known to be two schemes of resource allocation to users in SC-FDMA systems [1]. In localized scheme, a block of consecutive channels in the spectrum is allocated to each user. In the interleaved scheme channels will be spread out over the spectrum and will be distributed equidistantly. Performance comparison and system implementation of the schemes are investigated in the literature [1], [4]. Myuang et al. in [1] have found that localized SC-FDMA with channel-dependent scheduling (CDS) results in higher throughput than the interleaved one, while the PAPR performance of interleaved SC-FDMA is better than that of localized one. Song et al. in [4] conclude that interleaved SC-FDMA has less carrier frequency offset (CFO) interference, however localized scheme achieves more diversity gain.

There are various performance objectives such as maximum utility, minimum power, and minimum number of channels. Lei et al. [5] have proposed a graph labeling algorithm (GLA) for localized SC-FDMA with these objectives. They modeled SC-FDMA channel allocation through finding an optimal path in an acyclic graph. In the GLA, the concepts of labeling and label domination are applied to find the longest or shortest path. GLA is highly efficient in attaining close to optimal solutions. They proved localized SC-FDMA with objective minimum-power is NP-hard. As mobile users all use battery-powered portable devices, it is of significance to consider minimum sum power, subject to meeting a specified up-link demand target [6]-[8].

Water filling algorithm is a well-known method that has been applied on OFDM problems in the literature [9]-[13]. In this paper, when a block of channels is assigned to a user, a water-filling algorithm proved to be able to find the minimum power and then the algorithm is integrated with GLA proposed in [5]. Next, an algorithm for interleaved SC-FDMA with objective min-power is proposed to achieve global optimality in polynomial time. Finally, we compare their performances. The contributions are as follows:

- We propose a linear time water filling procedure OPA for computing optimal power allocation on channels.
- The power optimization procedure OPA is integrated with GLA, presented in [5].

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- We show that the minimum sum power for interleaved channel allocation scheme admits polynomial-time algorithm for global optimality either with or without the extension of power optimization.
- A polynomial-time algorithm MPI is developed to compute the minimum sum power for interleaved channel allocation scheme.
- Numerically, we find that by implementing power optimization procedure OPA less power is consumed.

This paper is organized as follows. In section II, we introduce Min-Power problem. In section III, we prove minimum power can be computed when a block of channels is allocated to a user, then algorithm's description and its pseudo-code are proposed. In section IV, we prove the Min-Power problem is polynomial solvable for interleaved channel allocation scheme. The algorithm's description and its pseudo-code are proposed as well. Numerical results are given in section V. Section VI concludes this paper.

II. Min-Power Problem

There are user set $\mathcal{M}=\{1,2,\ldots,M\}$ and channels set $\mathcal{N}=\{1,2,\ldots,N\}$. Each user is allocated with some channels. Each user has power limit P^u and each channel has peak power limit P^s. when a block of n channels is assigned to a user then consumed power on each channel should be less than or equal to min $\{P^u/n, P^s\}$. β is used to denote set of all possible blocks of channels. Each block of channels $b \in \beta$ is a set of consecutive channels or distributed equidistantly channels in localized and interleaved schemes respectively. A channel allocation corresponds to M mutually disjoint channels block $b_1, b_2 \ldots, b_M \in \beta$. Let $f(i,j,p_{ij})$ to denote the rate of user i on channel j with power level p_{ij} and $u_{ib} = \sum_{j\in b} f(i,j,p_{ij})$ to denote the rate of assigning block b to user i. Finally, user demand target is denoted by d_i for user i. Min- power problem can be stated as follows:

[Min-Power] Finds a channel-user allocation $b_1, b_2, \ldots, b_M \in \beta$ minimizing $\sum_{i\in M} p_{ib_i}$ where b_i is a feasible allocation for user i, for all $i \in \mathcal{M}$ and $b_i \cap b_j = \varnothing; \forall i, j \in \mathcal{M}$ or determine it is not feasible to meet the demands of all users within the power limits.

It has been assumed that a user has the same amount of power on the assigned channels i.e., $p_{ij}=p$ for all i,j [5,13]. With this assumption, minimization of f is straightforward by bi-section method. As the channels have different gains for different users, power can be used more efficiently by assigning users with channels that offer better gains. That is, for the given channel assignment to a user, its power on each of the assigned channels is subject to optimization as well. In this paper, we drop the assumption of equal-power distribution on channels and also remove the constraint of channel peak power (denoted by P^s), to allow for another dimension of freedom in power allocation. The considered rate-power function is $f(i,j,p_{ij}) = \log_2\left(1 + p_{ij} g_{ij} / \sigma^2\right)$ where p_{ij} and g_{ij} denote power and channel gain for user i on channel j respectively, and σ^2 is the noise power spectral density times the channel bandwidth B.

III. Optimal User-Channels Allocation

A. Problem Definition

User i and block b with n channels are given, the problem is to find optimal power allocation on each channel of block b for user i. The problem can be modeled as follows:

$$
\begin{aligned}
& p_{ib} = \min \sum_{j\in b} p_{ij} \\
& \text{s.t} \\
& \sum_{j\in b} \log_2(1 + c_{ij} p_{ij}) \ge d_i \\
& \sum_{j\in b} p_{ij} \le P^u \\
& p_{ij} \ge 0; \forall j \in b \\
& c_{ij} \text{ is given by } g_{ij} / \sigma^2
\end{aligned}
$$

This problem is equivalent to maximizing $\sum_{j\in b} \log_2(1 + c_{ij} p_{ij})$ when it is equal to d_i. The Power-rate function $\sum_{j\in b} \log_2(1 + c_{ij} p_{ij})$ is a convex function. For optimality, the derivative of the power-rate function on each channel has to be equal to each other. Since the function is convex, this condition is both necessary and sufficient. The derivative of function on each channel can be considered as the water level in water-filling algorithm where the water level must be equal on all channels. Suppose the channels are sorted by c_{ij} in the ascending order, the idea is to compute the optimal allocation of demand d_i on the channels denoted by d_{i1}, d_{i2}, $\ldots d_{in}$ with $\sum_j d_{ij} = d_i$. It is clear that if $d_{ij} = 0$, then d_{ik} for k>j are all zeros. In the first step, suppose we only have one channel, then obviously the entire demand d_i will be allocated on this channel, with $d_{i1}= d_i$. In the next step, suppose there are two channels. The task is to determine what fraction of d_i should be allocated to channel two. If the derivative of power-rate function at d_{i1} for channel one is bigger than the derivative of power-rate function at zero for channel two, then a fraction of d_i should be transferred to channel two. Assume d_{i1} and d_{i2} are allocated demand on channels one and two respectively, then:

$$
\begin{cases} \log_2(1+c_{i1}p_{i1})=d_{i1} \\ \log_2(1+c_{i2}p_{i2})=d_{i2} \end{cases} \tag{1}
$$

By rewriting (1) in exponential form, we have:

$$
\begin{cases} p_{i1} = \left(2^{d_{i1}} - 1\right) / c_{i1} \\ p_{i2} = \left(2^{d_{i2}} - 1\right) / c_{i2} \end{cases} \tag{2}
$$

At optimum $dp_{i1}/dd_{i1}=dp_{i2}/dd_{i2}$, then:

$$
2^{d_{i1}} / c_{i1} = 2^{d_{i2}} / c_{i2} \tag{3}
$$

By taking a logarithm in base 2, we have:

$$
d_{i1} = d_{i2} + \log_2(c_{i1} / c_{i2}) \tag{4}
$$

We also know $d_{i1}+d_{i2}=d_i$, thus by solving the following system:

$$
\begin{cases} d_{i1} = d_{i2} + \log_2(c_{i1} / c_{i2}) \\ d_{i1} + d_{i2} = d_i \end{cases} \tag{5}
$$

d_{i1} and d_{i2} can be determined and then p_{i1} and p_{i2} can be found easily. By repeating this process, both optimal demands and powers on all the n channels can be determined. If the power limits are not exceeded i.e., $\sum_{j\in b} p_{ij} \le P^u$, the allocation is feasible, otherwise the allocation is infeasible.

B. Linear-time Proof

In this section we describe algorithm "Optimal-Power-Allocation (OPA)" that finds optimal solution in linear time.

Theorem 1: Given user i and a block of channels b, optimal power allocation can be found in O(n).

Proof:

Consider OPA algorithm, total demand d is put on channel one in line 1. Line 2 will check the possibility of shifting any fraction of demand to the next channel. Lines 3-5 compute the fraction of demand on the channel when it is possible to shift any fraction of demand. Lines 6-7 will compute demand on the other channels. Finally, line 7 computes the consumed power on each channel.

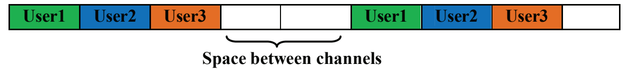
Line 1, lines 2- 5, lines 6-7 and line 8 can be run in O(1), O(n), O(n) and O(n) respectively. Therefore, complexity of OPA is O(1)+O(n)+O(n)+O(n)=O(n). ■

Algorithm : Optimal-Power-Allocation (OPA)
Input : user i, total demand d_i, coefficients c_{ij} for j=1,...,n
Output : optimal power p_{ij} for j=1,...,n
1 $j \leftarrow 1; d_{ij} \leftarrow d_i; temp \leftarrow 0;$
2 While $(j <= n-1) \& \left(\frac{2^{d_{ij}}}{c_{ij}} > \frac{1}{c_{i(j+1)}}\right)$
3 $\quad temp \leftarrow temp + j*\log_2(c_{ij}/c_{i(j+1)});$
4 $\quad d_{i(j+1)} \leftarrow (d - temp)/(j+1);$
5 $\quad j \leftarrow j+1;$
6 $d_{it} \leftarrow d_{ij} + \log_2(c_{it}/c_{ij});$ for t=1,...,j-1
7 $d_{it} \leftarrow 0;$ for t=j+1,....,n
8 $p_{ij} \leftarrow \left(2^{d_{ij}} - 1\right)/c_{ij}$ for j=1,...,n
9 return p_{ij} for j=1,...,n

IV. Complexity of Min-Power Problem with Interleaved Scheme

In this section we prove that the Min-Power problem with interleaved channel allocation scheme admits polynomial-time algorithm for global optimality.

Consider N channels and M users, we assume that all users are allocated with the same number of channels. Then, each user can be provided with zero to [N/M] number of channels. If each user is allocated with zero channel then obviously the solution will be infeasible. Therefore, we start by allocating one channel to each user where the total number of choices is N-M+1. If each user is allocated with more than one channel, say L channels, then possible choices with interspace zero to $\left[\frac{N-LM}{L-1}\right]$ are given in table *I*. An example is shown in Fig. 1. Total number of these choices is:



Number of channels is 9,
Number of users is 3,
Each user is allocated with two channels, they are equidistant.
Interspace for this example is two channels.

Fig. 1. An example for interleaved channel allocation

TABLE I
TOTAL NUMBER OF CHOICES IN WHICH EACH USER GET L CHANNELS

Number of channels between blocks (space between blocks)	Number of choices
0	N-LM+1
1	N-LM-(L-1) × 1+1
2	N-LM-(L-1) × 2+1
3	N-LM-(L-1) × 3+1
. . .	. . .
$\left[\frac{N-LM}{L-1}\right]$	$N-LM-(L-1)\times\left[\frac{N-LM}{L-1}\right]+1$

$$(N-LM)\left(\left[\frac{N-LM}{L-1}\right]+1\right)-(L-1)\left(1+2+\dots+\left[\frac{N-LM}{L-1}\right]\right)+\left[\frac{N-LM}{L-1}\right]+1 \quad (6)$$

$$=(N-LM)\left(\left[\frac{N-LM}{L-1}\right]+1\right)-(L-1)\frac{\left[\frac{N-LM}{L-1}\right]\left(\left[\frac{N-LM}{L-1}\right]+1\right)}{2}+\left[\frac{N-LM}{L-1}\right]+1 \quad (7)$$

$$=\left(\left[\frac{N-LM}{L-1}\right]+1\right)\left(N-LM-\frac{(L-1)\left[\frac{N-LM}{L-1}\right]}{2}+1\right) \quad (8)$$

since there are [N/M] –1 cases for L then totally we have:

$$([N/M]-1)\left(\left[\frac{N-LM}{L-1}\right]+1\right)\left(N-LM-\frac{(L-1)\left[\frac{N-LM}{L-1}\right]}{2}+1\right) \quad (9)$$

The following algorithm "Compute Channel Subsets (CCS)", compute all possible subsets.

Procedure-Compute Channel Subsets(CCS)
Input: M,N
Output: channel Subsets SB_i
$i \leftarrow 1;$
for L=1 to N-M+1
$\quad SB_i \leftarrow channels\{L,\dots,M+L-1\};$
$\quad i \leftarrow i+1;$
for L=2 to [N/M]
$\quad$ for j=0 to $\left[\frac{N-LM}{L-1}\right]$
$\quad\quad h \leftarrow \{\};$
$\quad\quad$ for s=1 to L
$\quad\quad\quad h \leftarrow h \cup channels\{1+(s-1)M+j,\dots,sM+j\};$
$\quad\quad SB_i \leftarrow h;$
$\quad\quad i \leftarrow i+1;$
$\quad\quad$ for i=1 to N-LM-(L-1)j
$\quad\quad\quad SB_i \leftarrow h+i;$
$\quad\quad\quad i \leftarrow i+1;$
return $SB_1, SB_2, \dots, SB_i$

In line 1, “i” is used for counting number of subsets. Lines 2-4 compute all subsets with one channel for each user in which interspace cannot be defined. Lines 5-11 compute a subset with L channels for each user and interspace j. By shifting the subset found in lines 5-11, lines 12-14 compute all other possible subsets with L channels for each user and interspace j.

Theorem 2: with interleaved channel allocation scheme, all possible channel subsets can be computed in $O(N^3)$ provided that all users are allocated with the same number of channels**.**

Proof

Consider procedure CCS, Line 1, lines 2-4, and lines 5-14 run in $O(1)$, $O(N)$, and $O(N^3)$ respectively. Therefore, complexity of CCS is $O(1)+O(N)+O(N^3) = O(N^3)$. ■

In each case SB_i returned by CCS, there is a group of channels. Let $|SB_i|$ denotes number of these channels. Then each user will be assigned with a block b of channels that contains $|SB_i|/M$ ($|SB_i|/M \leq [N/M]$) channel(s) i.e., $|b|=|SB_i|/M$. If location of the first assigned channel to a user is known then location of other channels can be determined automatically as they are equidistant. Allocating M blocks to M users is equivalent to an assignment problem that can be solved by Hungarian method in polynomial time $O(M^3)$. Optimum solution can be found by following procedure “Min-Power for Interleaved scheme (MPI)”:

```
Procedure: Min – Power for Interleaved scheme(MPI)
1  MinPower ← ∞
2  Compute Channels Subsets (M,N)
3  for j=1 to i
4    subdivide SB_j to M blocks;
5    for u=1 to M
6       for b=1 to M
7          PowerMatrix(i,u,b) ← total consumed power on block b by user u;
8  for j=1 to i
9  MinPower ← min (MinPower, HungarianMethod(PowerMatrix(i)));
10 return MinPower
```

The algorithm MPI will compute minimum power. In line 1, MinPower is used for saving the best so far solution and initially is set to infinity. Line 2 computes all possible subsets by procedure CCS. In Lines 3-7, each subset will be subdivided into M blocks in which channels are equidistant. Then consumed power on each block will be computed either by OPA when we use the power optimization OPA or Bisection/Newton methods when equal-power distribution on channels is assumed. Lines 8-9 will find optimum solution by Hungarian method and the best so far solution will be saved in MinPower.

Theorem 3: For interleaved channel allocation, the Min-Power problem admits polynomial-time algorithm for global optimality with or without the extension of power optimization.

Proof

We just need to show that MPI runs in polynomial time: Line 1, line 2, lines 3-7, and lines 8-9 run in $O(1)$, $O(N^3)$, $O(iM^2|b|)$, and $O(iM^3)$ respectively. Therefore complexity of MPI is:

$$O(1)+O(N^3)+O(iM^2|b|)+O(iM^3) \tag{10}$$

we know that $i=O(N^3)$ and $|b|=|SB_i|/M \leq N/M$, then

$$O(1)+O(N^3)+O(iM^2|b|)+O(iM^3) \leq \tag{11}$$

$$O(1)+O(N^3)+O(N^4M)+O(N^3M^3) = \tag{12}$$

$$O(N^3)+O(N^4M)+O(N^3M^3) = \tag{13}$$

$$O(N^4M^3). \blacksquare \tag{14}$$

V. Performance Comparison

For performance evaluation, we consider SC-FDMA uplink of a cell with randomly and uniformly distributed users. Table *II* summarizes the key parameters. The channel gain consists of path loss, shadowing, as well as Rayleigh fading. The path loss follows the widely used COST 231 model that extends the Okumura-Hata model for urban scenarios. By the COST 231 model, path loss is frequency dependent. Log-normal shadowing model with 8 dB standard deviation is used [14]. A channel corresponds to a resource block in LTE with twelve subcarriers. We integrate OPA with GLA proposed in [5] for two instances G1 and G2. Next, we implement MPI algorithm once with the assumption of equal power on channels, then with OPA. The results are shown in table *III*.

The result from localized scheme seems to be slightly greater in terms of its performance when it is compared to the interleaved one. However, the localized and interleaved schemes with OPA showed a better result when they are compared to findings shown on the localized and interleaved without OPA. Furthermore, the interleaved with power optimization demonstrated the preferable level of power compare to the localized scheme without power optimization. Finally, an outstanding finding for both instances was thought to be localized scheme with OPA when it is compared to all other schemes.

TABLE II
SIMULATION PARAMETERS

parameter	value
Cell radius	1000 m
Carrier frequency	2 Ghz
Number of users M	10
Number of channels N	64
Channel bandwidth B	180
Path loss	COST-23-KHz-HATA
shadowing	Log-normal, 8dB standard deviation
Multipath fading	Rayleigh fading
Noise power spectral density	-174 dBm/Hz
User power limitP^u	200 mW
Chanel peak power limit P^s	10 mW
Demand for each user d_i	400 Kbps

TABLE III
PERFORMANCE COMPARISON

Instance	Localized scheme	Localized scheme with OPA
G1 10×64	0.0170472	0.0142092
G2 10×64	0.0308121	0.0258781

Instance	Interleaved scheme	Interleaved scheme with OPA
G1 10×64	0.0161999	0.0152138
G2 10×64	0.0300472	0.0270982

VI. Conclusions

In this paper, the Min-Power problem with localized and interleaved channels allocation schemes is considered. The assumption of equal-power distribution on the assigned channels dropped to allow for another dimension of freedom in power allocation. It has been proved that for interleaved channel allocation, the Min-Power problem admits polynomial-time algorithm for global optimality with or without the extension of power optimization, By implementing the proposed algorithms, localized scheme with the power optimization OPA outperform all the other schemes. In addition, it has been seen that the interleaved SC-FDMA with this new assumption has a better performance.

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APPENDIX

GLA alghorithm for Min-Power is as follows:

The GLA algorithm for Min-Power

Input: $\mathcal{M}, \mathcal{N}, \mathcal{B}, u_{ib}, i \in \mathcal{M}, b \in \mathcal{B}$
Output: ℓ^* (the best label value)

for $j = 0 : N$ **do**
  $\mathcal{L}(j, 0) \leftarrow \{(0, 0, \emptyset)\}$
  for $m = 1 : M$ **do**
    $\mathcal{L}(j, m) \leftarrow \emptyset$
for $j = 1 : N$ **do**
  for $\min\{j, M\}: m = 1$ **do**
    for $i = 0 : j - 1$ **do**
      $\mathcal{L}' \leftarrow \mathcal{L}(i, m)$
      for all $\ell \in \mathcal{L}(i, m - 1)$ **do**
        for all $w \notin \mathcal{M}_\ell$ **do**
          $\mathcal{L}' \leftarrow \mathcal{L}' \cup \{(v_\ell + v^w_{i+1,j}, M_\ell + 1, \mathcal{M}_\ell \cup \{w\})\}$
      for all $\ell' \in \mathcal{L}'$ **do**
        if $\nexists h \in \mathcal{L}_{js}, s = m, \ldots, \min\{j, M\}$:
          $v_{\ell'} \geq v_h \wedge \mathcal{M}_{\ell'} \subseteq \mathcal{M}_h$ **then**
          if $|\mathcal{L}(j, m)| < K$ **then**
            $\mathcal{L}(j, m) \leftarrow \mathcal{L}(j, m) \cup \{\ell'\}$
          else
            $\bar{\ell} \leftarrow \text{argmax}_{h \in \mathcal{L}(j,m)} v_h$
            if $v_{\bar{\ell}} > v'_\ell$ **then**
              $\mathcal{L}(j, m) \leftarrow \mathcal{L}(j, m) \setminus \{\bar{\ell}\}$
              $\mathcal{L}(j, m) \leftarrow \mathcal{L}(j, m) \cup \{\ell'\}$
$\ell^* \leftarrow \text{argmin}_{\in \mathcal{L}(N,m), m=1,\ldots,M} v_\ell$
return ℓ^*