Best Error Bounds for Splines of Degree Seven

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ABSTRACT

In this paper, we construct a spline method to solving a interpolation problem using spline polynomial of degree seven which agree with the given function and it's second derivative at knots and the function at mid points and the second derivative at (1/3) points also. This new class of spline interpolates provides a large accuracy in the choice of the error bounds.

1. INTRODUCTION

The theory of spline functions and their applications are relatively recent development, classes of spline functions possess many nice structural properties as well as excellent approximation powers, since they are easy to store as (Ahlberg 1967 and Al-Bayati and et al 2009). The interpolation spline functions can be designed of curves and surfaces, in this way, the spline polynomial provides more smoothly and flexibility in the choice of an interpolation like as (Ahlberg 1967 and Karwan and Faraidun 2012). It plays an important role not only in the constructions of different produces such as car bodies, ship hulls, airplane fuse larges and wings, data fitting, function approximation, numerical quadrature and the numerical solution of operator equations such as these associated with ordinary and partial differential equations, integral equations, physical and even medical phenomena and so

on. Many researchers used different degree of spline functions of the type cubic, quadratic, quantic, and sixtic for different constructions as (El Tarazi and Sallam 1990, François an et al 1985, John 2005, Meyer and Hall 1976, Sallam and Anwar 1999), and also they obtained the error bounds for each cases.

Researchers have investigated lacunary spline interpolation in the last 50 years. Varma 1973 and 1978, Ahlberg et al 1967 and Albayati et al 2009, developed a fundamental approximation theory and proved existence and uniqueness for constructed the spilne functions. Rana and Dubey 1997, Rana and et al 2011, showed that the best error bounds of deficient quantic spline interpolation. There are some of the earliest studies on spline polynomials and lacunary interpolations in (Ezio 1996 and Karwan and Faraidun 2012).

In this paper, the main objective is to construct a seven degree spline function to develop numerical technique that are possible

for interpolating functions, which is based on fact that the spline function can be interpolated knots on a set of function. We proof that the constructed seven degree spline polynomial supported functions for a basis of best approximation, furthermore properties of functions are given and error bounds and briefly discuss its convergence analysis.

2. Existence and Uniqueness

Let Δ denote a gird or partition of the unit interval $I = [0,1]$, i.e

$$\Delta: 0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$$

The maximal length is abbreviated by
$$h = h(\Delta) = \max_{0 \leq i \leq n-1} h_i, h_i = x_{i+1} - x_i, i = 0,1,..,n-1$$

As usual by $S_p(7,\Delta)$, we denote the space of deficient seven degree spline functions determined by the above partition Δ, which are developed from (Rana and Dubey 1997, Rana and et al 2011) respectively, and this model namely $S \in S_p(7,\Delta)$ if and only if the following conditions are satisfied:

1) in each subinterval $[x_i, x_{i+1}], i = 0,1,...,n-1$, S is a polynomial of degree seven or less
2) $S \in C^4(I)$

Now when $f \in C^8(I)$ is a given function, we are intersected in condition under which there exist a unique deficient seven degree spline which satisfies the following interpolating conditions:

$$S(x_i) = f(x_i), S''(x_i) = f''(x_i), i = 0, 1, 2,..., n$$

and
$$S(z_i) = f(z_i), S''(w_i) = f''(w_i), i = 0,1,...,n-1$$

Where $w_i = x_i + \left(\frac{1}{3}\right)h_i$ and $z_i = x_i + \left(\frac{1}{2}\right)h_i$,

$$(2.1)$$

and boundary conditions

$$S'''(x_0) = f'''(x_0), S'''(x_n) = f'''(x_n) \quad (2.2)$$

We now state the following:

Theorem 2.1: For given functional values and derivatives $f(x_i), f''(x_i), i = 0,1,2,...,n$ and $f(z_i), f''(w_i), i = 0,1,...,n$-1 along with $f'''(x_0), f'''(x_n)$ there exists a unique $S \in S_p(7,\Delta)$ which satisfies conditions (2.1) and (2.2).

Let $P(x)$ be a polynomial of degree seven on [0, 1], then it is easy to verify that

$$P(x) = P(0)A_0(x) + P\left(\frac{1}{2}\right)A_1(x) +$$
$$P(1)A_2(x) + P''(0)A_3(x) + P''\left(\frac{1}{3}\right)A_4(x) +$$
$$P''(1)A_5(x) +$$
$$P'''(0)A_6(x) + P'''(1)A_7(x), \quad (2.3)$$

where
$$A_0(x) = \frac{1}{77}(77 - 141x - 1120x^4 + 3360x^5 - 3136x^6 + 960x^7)$$
$$A_1(x) = \frac{1}{77}(128x + 2240x^4 - 6720x^5 + 6272x^6 - 1920x^7)$$
$$A_2(x) = \frac{1}{77}(13x - 1120x^4 + 3360x^5 - 3136x^6 + 960x^7)$$
$$A_3(x) = \frac{1}{770}(-107x + 385x^2 - 1680x^4 + 2807x^5 - 1855x^6 + 450x^7)$$

$$A_4(x) = \frac{1}{560}(-81x + 2835x^4 - 6804x^5 + 5670x^6 - 1620x^7)$$
$$A_5(x) = \frac{1}{6160}(-53x + 4655x^4 - 14812x^5 + 15190x^6 - 4980x^7)$$
$$A_6(x) = \frac{1}{4620}(-32x + 770x^3 - 2485x^4 + 3297x^5 - 2030x^6 + 480x^7)$$

$$(2.4)$$

$$A_7(x) = \frac{1}{9240}(9x - 805x^4 + 2646x^5 - 2870x^6 + 1020x^7)$$

.

Proof of Theorem 2.1:

Now writing $x = x_i + th_i, 0 \leq t \leq 1$ we denote by S_i the restriction of the spline S to $[x_i, x_{i+1}]$ and using the represented of (2.3)-(2.4), we may express,

$$S_i(x) = f(x_i)A_0(t) + f(z_i)A_1(t) + f(x_{i+1})A_2(t) + h_i^2 f''(x_i)A_3(t) + h_i^2 f''(w_i)A_4(t)$$

$$+h_i^2 f''(x_{i+1})A_5(t) + h_i^3 S_i'''(x_i)A_6(t) +$$
$$h_i^3 S_t'''(x_{i+1})A_6(t)$$
$$\tag{2.5}$$

Since $S \in C^4(I)$, we have from (2.5),

$$840h_i S'''(x_{i-1}) + (6090h_i +$$
$$5964h_{i-1})S'''(x_i) + 966h_{i-1}S'''(x_{i+1}) = F_i$$
, i=1, 2,…, n-1 $\tag{2.6}$

Where

$$F_i = 6f(x_i)[-\frac{26880}{h_i^3}h_{i-1} - \frac{53760}{h_{i-1}^3}h_i] +$$
$$6[\frac{53760}{h_i^3}h_{i-1}f(z_i) + \frac{107520}{h_{i-1}^3}h_i f(z_{i-1})]$$

$$6[-\frac{26880}{h_i^3}h_{i-1}f(x_{i+1}) - \frac{53760}{h_{i-1}^3}h_i f(x_{i-1})]$$
$$+3f''(x_i)[-\frac{8064}{h_i}h_{i-1} + \frac{9513}{h_{i-1}}h_i]+33[$$
$$\frac{1701}{h_i}h_{i-1}f''(w_i)$$

$$+\frac{1701}{h_{i-1}}h_i f''(w_{i-1})]+3$$
$$[\frac{2793}{h_i}h_{i-1}f''(x_{i+1}) - 1344\frac{h_i}{h_{i-1}}f''(x_{i-1})].$$

Clearly the system of equations (2.6) is diagonally dominant and hence the coefficient matrix of the system of equation (2.6) is invertible, which completes the proof of theorem 2.1.

3. Convergence and Error Analysis

Following the method of (Meyer and Hall 1976), in this section we estimate the bounds of error, $e(x) = f(x) - S(x)$ for spline interpolation of theorem 2.1 which are best.

Theorem 3.1: suppose that $s(x)$ is the seventic spline of theorem 2.1 interpolating a function $f(x)$ and $f \in C^8[0,1]$, then

$$e(x) \leq m \frac{h^8}{8!} \max_{0 \leq x \leq 1} |f^{(8)}(x)| \tag{3.1}$$

Where $m = \max_{0 \leq t \leq 1} |M(t)|,$ and

$$M(t) = \left[t^3(1-t)^3 \left(\frac{1}{2}-t\right)\left(\frac{1}{3}-t\right) + \frac{k}{1086}k(t)\right]$$

Also we have

$$e(x_i) \leq k_1 \frac{h^8}{8!} \max_{0 \leq x \leq 1} |f^{(8)}(x)|, \quad k_1 = \frac{k_0}{1086},$$
where $k_0 = \frac{1002080296}{10}$. $\tag{3.2}$

Proof: Let $S(x)$ be third time continuously differentiable seventic spline function satisfy the conditions of theorem 2.1. Now considering $f \in C^{(8)}[0,1]$ and writing $L_i[f,x]$ for the unique seventic which agress with $f(x_i), f(z_i), f(x_{i+1}), f''(x_i), f''(x_{i+1})$, $f''(w_i)$, we see that for $x \in [x_i, x_{i+1}]$, we have
$$|f(x) - S(x)| \cong |f(x) - S_i(x)| \leq | S_i(x) - L_i[f,x]| + | f(x) - L_i[f,x]|$$
$$\tag{3.3}$$

In order to obtain the bounds of $e(x)$, we proceed to get pointwise bounds of both the term on the right hand side of (3.3). The estimate of the first term can be obtained by following a well-known theorem of Cauchy in Davis 1961, i.e.

$$\| f(x) - L_i[f,x] \| \le \frac{h_i^8}{8!} \left| t^3(1-t)^3 \left(\frac{1}{2} - t\right)\left(\frac{1}{3} - t\right) \right| F,$$

(3.4)

Where $t = \frac{x-x_i}{h_i}$ and $F = \max\limits_{0 \le x \le 1} |f^{(8)}(x)|$.

To get the bounds of $\| S_i(x) - L_i[f,x] \|$, we have from (2.3)

$$S_i(x) - L_i[f,x] = h_i^3 \{ [e^{'''(x_i)}]A_6(t) + [e^{'''(x_{i+1})}]A_7(t) \}$$

(3.5)

Thus,

$$\| S_i(x) - L_i[f,x] \| \le h_i^3 \{ |e^{'''}(x_i)| |A_6(t)| + |e^{'''}(x_{i+1})| |A_7(t)| \}$$

(3.6)

Noted that

$$|A_6(t) + A_7(t)| = \left| \frac{1}{4620}(-32t + 770t^3 - 2485t^4 + 3297t^5 - 2030t^6 + 480t^7) + \frac{1}{9240}(9t - 805t^4 + 2646t^5 - 2870t^6 + 1020t^7) \right| = \frac{1}{9240} |(-64t + 1540t^3 - 4970t^4 + 6594t^5 - 4060t^6 + 960t^7) + (9t - 805t^4 + 2646t^5 - 2870t^6 + 1020t^7)| = \frac{1}{9240} |(-55t + 1540t^3 - 5775t^4 + 9240t^5 - 6930t^6 + 1980t^7)| = k(t)$$

$$|A_6(t) + A_7(t)| = \frac{1}{9240} |(-55t + 1540t^3 - 5775t^4 + 9240t^5 - 6930t^6 + 1980t^7)| = k(t), \text{ where } 0 \le t \le 1$$

(3.7)

By applying (3.7) in (3.6) we get that

$$\| S_i(x) - L_i[f,x] \| \le h_i^3 \max\{|e^{'''}(x_i)|, |e^{'''}(x_{i+1})|\} k(t)$$

(3.8)

Let the $\max\limits_{0 \le j \le n} |e^{'''}(x_j)|$ exists for i, then (3.6) may be written as

$$\| S_i(x) - L_i[f,x] \| \le h_i^3 |e^{'''}(x_i)| k(t)$$

(3.9)

Replacing $S^{'''}(x_j)$ by $e^{'''}(x_j)$ in (2.6), we have

$$840h_i e^{'''}(x_{i-1}) + (6090h_i + 5964h_{i-1})e^{'''}(x_i) + 966h_{i-1}e^{'''}(x_{i+1}) = C(f),$$

(3.10)

where

$$C(f) = 6f(x_i)\left[-\frac{26880}{h_i^3}h_{i-1} - \frac{53760}{h_{i-1}^3}h_i \right] + 6\left[\frac{53760}{h_i^3}h_{i-1}f(z_i) + \frac{107520}{h_{i-1}^3}h_i f(z_{i-1}) \right] +$$

$$6[-\frac{26880}{h_i^3}h_{i-1}f(x_{i+1}) - \frac{53760}{h_{i-1}^3}h_i f(x_{i-1})]$$

$$+3f''(x_i)[-\frac{8064}{h_i}h_{i-1} + \frac{9513}{h_{i-1}}h_i]+33$$

$$[\frac{1701}{h_i}h_{i-1}f''(w_i)$$

$$+\frac{1701}{h_{i-1}}h_i f''(w_{i-1})]+3$$

$$[\frac{2793}{h_i}h_{i-1}f''(x_{i+1}) - 1344\frac{h_i}{h_{i-1}}f''(x_{i-1})]$$

$$-840h_i f^{'''}(x_{i-1})-$$

$$(6090h_i + 5964h_{i-1})f^{'''}(x_i) - 966h_{i-1}f^{'''}(x_{i+1})$$

(3.11)

Since $C(f)$ is a linear function which is zero for polynomial of degree seven or less, we can apply the Peano theorem [3] to get

$$C(f) = \int_{x_{i-1}}^{x_{i+1}} \frac{f^{(8)}(y)}{7!} C[(x-y)_+^7] dy$$

(3.12)

From (3.9) it follows that

$$|C(f)| \le \frac{F}{7!} \int_{x_{i-1}}^{x_{i+1}} |C[(x-y)_+^7]| dy$$

(3.13)

Further, it can be observe from (3.9) that for $x_{i-1} \le y \le x_{i+1}$

$$C[(x-y)_+^7] = 6(x_i - y)_+^7 \left[-\frac{26880}{h_i^8} h_{i-1} - \frac{53760}{h_{i-1}^8} h_i \right] + 6[\frac{53760}{h_i^8} h_{i-1}(z_i - y)_+^7 + \frac{107520}{h_{i-1}^8} h_i(z_{i-1} - y)_+^7]$$

$$-\frac{161280}{h_i^8} h_{i-1}(x_i - y)_+^7 + 126(x_i - y)_+^5 \left[-\frac{8064}{h_i} h_{i-1} + \frac{9513}{h_{i-1}} h_i \right] + 1386$$

$$[\frac{1701}{h_i} h_{i-1}(w_i - y)_+^5 + \frac{1701}{h_{i-1}} h_i(w_{i-1} - y)_+^5] + 3\frac{351918}{h_i} h_{i-1}(x_{i+1} - y)_+^5 -$$
$$210(6090h_i + 5964h_{i-1})(x_i - y)_+^4 - 202860h_{i-1}(x_{i+1} - y)_+^4$$

Rewriting the above expression in the following symmetric form about x_j we get

$$C[(x-y)_+^7] = \begin{cases} f_1(y), x_{i-1} \le y \le w_i \\ f_2(y), \ w_i \le y \le z_i \\ f_3(y), \ z_i \le y \le x_i \\ f_4(y), \ x_i \le y \le x_{i+1} \end{cases}$$

(3.14)

When

$$f_1(y) = \left[-\frac{322560}{h_{i-1}^8} h_i(x_i - y)^7 + \frac{1128960}{h_i^2} h_{i-1}(x_i - y)^6 + \left[\frac{4090716}{h_i} h_{i-1} + \frac{1198638}{h_{i-1}} h_i \right] (x_i - y)^5 + [9163980h_{i-1} - 1278900h_i](x_i - y)^4 + 13071240h_{i-1}h_i(x_i - y)^3 + 10425240h_{i-1}h_i^2(x_i - y)^2 + 4648140h_{i-1}h_i^3(x_i - y) + 865116h_{i-1}h_i^4 \right]$$

$$f_2(y) = \left[-\frac{322560}{h_{i-1}^8} h_i(x_i - y)^7 + \frac{1128960}{h_i^2} h_{i-1}(x_i - y)^6 + \left[\frac{1733130}{h_i} h_{i-1} + \frac{1198638}{h_{i-1}} h_i \right] (x_i - y)^5 + [5234670h_{i-1} - 1278900h_i](x_i - y)^4 + 10451700h_{i-1}h_i (x_i - y)^3 + 9552060h_{i-1}h_i^2(x_i - y)^2 + 4502610h_{i-1}h_i^3(x_i - y) + 855414h_{i-1}h_i^4 \right]$$

$$f_3(y) = \left[-\frac{322560}{h_i^8} h_{i-1}(x_i - y)^7 - \frac{322560}{h_{i-1}^8} h_i(x_i - y)^7 + \left[\frac{39690}{h_i} h_{i-1} + \frac{1198638}{h_{i-1}} h_i \right] (x_i - y)^5 + [3823470h_{i-1} - 1278900h_i] (x_i - y)^4 + 9746100h_{i-1}(x_i - y)^3 h_i + 9340380h_{i-1}(x_i - y)^2 h_i^2 + 4467330h_{i-1}(x_i - y) h_i^3 + 852894h_{i-1}h_i^4 \right]$$

$$f_4(y) = \left[\frac{1055754}{h_i} h_{i-1}(x_i - y)^5 + 5075910h_{i-1}(x_i - y)^4 + 9746100h_{i-1}(x_i - y)^3 h_i + 9340380h_{i-1}(x_i - y)^2 h_i^2 + 4467330h_{i-1}(x_i - y) h_i^3 + 852894h_{i-1}h_i^4 \right]$$

And it is obvious from the above expression that $C[(x-y)_+^7]$ is non negative for $x_{i-1} \le y \le x_{i+1}$, Therefore it follows that

$$\int_{x_{i-1}}^{x_{i+1}} |C[(x-y)_+^7]| dy = \frac{315}{2h_{i-1}^8} (h_i)^9 + \frac{1128960}{7h_i^2} (h_{i-1})^8 + \frac{4090716}{h_i} \frac{1}{6} (h_{i-1})^7 + 163980\frac{1}{5} (h_{i-1})^6 + \frac{142352}{10} h_{i-1}(h_i)^5 + 3171483h_i(h_{i-1})^5 + \frac{10425240}{3} h_i^2 (h_{i-1})^4 + \frac{4648140}{2} h_i^3 (h_{i-1})^3 + 65116h_i^4 (h_{i-1})^2$$

(3.15)

Combining (3.12) with (3.15), we get

$$|C(f)| \leq \frac{F}{8!}\left[\frac{1260}{h_{i-1}^3}(h_i)^9 + \frac{1290240}{h_i^2}(h_{i-1})^8 + \frac{5454288}{h_i}(h_{i-1})^7 + 14662368(h_{i-1})^6 + \frac{1138816}{10}h_{i-1}(h_i)^5 + 25371864h_i(h_{i-1})^5 + 27800640h_i^2(h_{i-1})^4 + 18592560h_i^3(h_{i-1})^3 + 6920928h_i^4(h_{i-1})^2\right]$$

(3.16)

Then by using (3.9), (3.10) and (3.16), we obtain

$$|e'''(x_i)| \leq \frac{F}{(3930 h_i + 6930 h_{i-1})8!}\left[\frac{1260}{h_{i-1}^3}(h_i)^9 + \frac{1290240}{h_i^2}(h_{i-1})^8 + \frac{5454288}{h_i}(h_{i-1})^7 + 14662368(h_{i-1})^6 + \frac{1138816}{10}h_{i-1}(h_i)^5 + 25371864h_i(h_{i-1})^5 + 27800640h_i^2(h_{i-1})^4 + 18592560h_i^3(h_{i-1})^3 + 6920928h_i^4(h_{i-1})^2\right]$$

(3.17)

Also by using (3.4), (3.8) along (3.13) in (3.3) we get

$$e(x) \leq \frac{h^8}{8!}F|M(t)|$$

(3.18)

Where

$$M(t) = \left[t^3(1-t)^3\left(\frac{1}{2}-t\right)\left(\frac{1}{3}-t\right) + \frac{k_0}{1086}k(t)\right], \quad k_0 = \frac{1002080296}{10}$$

Thus we can proved the result of the theorem.

Note: Furthermore, k in (3.1) cannot be improved for an equality spaced partition. Inequality (3.18) is also best possible. Now, we turn to see that the form of inequality (3.2) in the limit. Considering $f(x) = \frac{x^8}{8!}$ and using the Cauchy formula in Davis 1961, we have

$$L_i\left[\frac{t^8}{8!}, x\right] - \frac{x^8}{8!} = \frac{h_i^8}{8!}\left(t^3(t-1)^3\left(t-\frac{1}{2}\right)\left(t-\frac{1}{3}\right)\right)$$

(3.19)

Moreover for the function under consideration (3.10) gives the following for equally spaced knots

$$C\left(\frac{x^8}{8!}\right) = k_0\frac{h^5}{8!} = 840e'''(x_{i-1}) + 12054e'''(x_i) + 966e'''(x_{i+1})$$

(3.20)

Considering for a moment

$$e'''(x_i) = k_1\frac{h^5}{8!} = e'''(x_{i-1}) = e'''(x_{i+1})$$

(3.21)

We have from (3.4)

$$s(x) - L_i[f,x] = k_1\frac{h^8}{8!}\{A_6(t) + A_7(t)\} = k_1\frac{h^8}{8!}k(t)$$

(3.22)

Combining (3.19) and (3.22) we get for $x_i \leq x \leq x_{i+1}$

$$s(x) - \frac{x^8}{8!} = \frac{h^8}{8!}\left[k_1k(t) + \left(t^3(t-1)^3\left(t-\frac{1}{2}\right)\left(t-\frac{1}{3}\right)\right)\right]$$

(3.23)

From (3.23) it is clearly observed that (3.2) is best possible provided we could prove that

$$e'''(x_i) = e'''(x_{i-1}) = e'''(x_{i+1}) = k_1\frac{h^5}{8!}$$

(3.24)

In fact (3.24) is attained only in the limit. The difficulty will take place in the case of boundary condition i.e. $e'''(x_0) = e'''(x_n) = 0$. However, it can be shown that as we move many subintervals away from the boundaries,

$$e'''(x_i) \to k_1 \frac{h^5}{8!}$$

For that we shall apply (3.20) inductively to move away from the end conditions,

$$e'''(x_0) = e'''(x_n) = 0.$$

The first step in this direction is to show that $e'''(x_i) \geq 0$, for $i = 0, \ldots, n$ which can be proved by a contradictory assumption.

Let $e'''(x_i) < 0$, for some $i, i = 1, \ldots, n - 1$. Now making a use of (3.17), we get

$$k_1 \frac{h^5}{8!} \geq \max |e'''(x_i)| \geq \frac{1}{10} \{840 e'''(x_{i-1}) +$$
$$966 e'''(x_{i+1})\} > \frac{1}{10} \{840 e'''(x_{i-1}) +$$
$$12054 e'''(x_i) + 966 e'''(x_{i+1})\} = k_0 \frac{h^5}{8!}$$

Which is a contradiction, therefore $e'''(x_i) \geq 0$, for $i = 0, \ldots, n$, now from (3.20) we can write

$$12054 e'''(x_i) = k_0 \frac{h^5}{8!} - \{840 e'''(x_{i-1}) +$$
$$966 e'''(x_{i+1})\}$$

And since $e'''(x_i) \geq 0$, so we get

$$e'''(x_i) \leq \frac{k_0}{12054} \frac{h^5}{8!} , for \ i = 0, \ldots, n \quad (3.25)$$

Now again using (3.25) in (3.20) we get

$$e'''(x_i) \leq \frac{k_0}{12054} \frac{h^5}{8!} \left[1 - \frac{1}{12054} \right] i = 0, \ldots, n,$$

thus a repeat of (3.20), we obtain

$$e'''(x_i) \leq \frac{k_0}{12054} \frac{h^5}{8!} \left[1 - \frac{1}{12054} + \left(\frac{1}{12054} \right)^2 - \right.$$
$$\left. \cdots + \left(\frac{1}{12054} \right)^{i-1} \right]$$

$$(3.26)$$

Now it can be seen easily that the right hand side of (3.25) $\to k_1 \frac{h^5}{8!}$ and hence in the limiting case

$$e(x_i) \leq k_1 \frac{h^8}{8!} \tag{3.27}$$

Which verifies proof of (3.18), corresponding to $f(x) = \frac{x^8}{8!}$ and for equally spaced knots in the limit. This proves theorem 3.1 completely.

4. Conclusion

In this paper, we have studied the existence and uniqueness of a class of spline function of degree seven that match the derivatives at the knots to a given order. Also best error bounds for certain interpolation polynomial formula based on the values of the derivatives of the spline seven degree and convergence analysis derived.

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